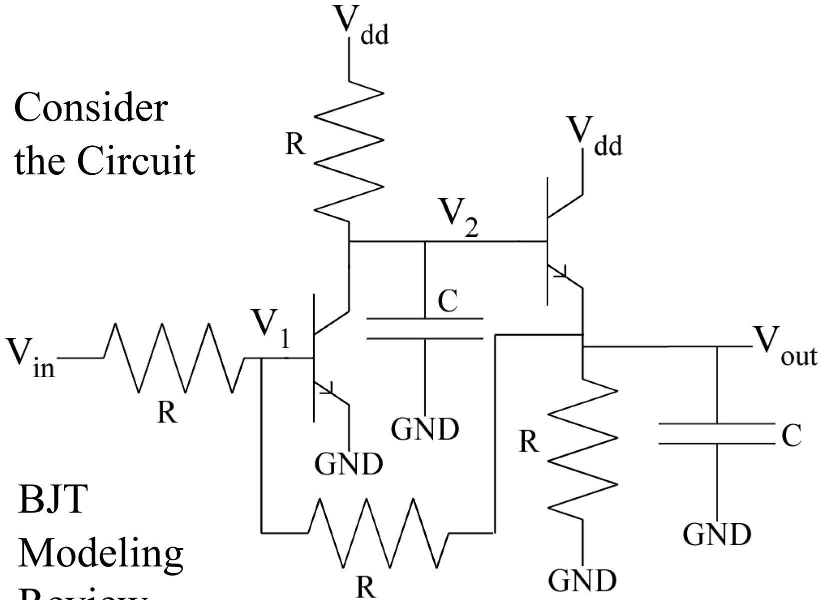
Linearizing a Circuit Around a Steady State

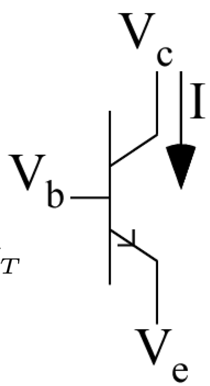
Consider the Circuit



BJT Modeling Review

$\beta \to \infty$

Only one current (no base current)



$$I_c = I_s e^{(V_b - V_e)/U_T} = I_s e^{(V_{be})/U_T}$$

$$C\frac{dV_2}{dt} = \frac{V_{dd} - V_2}{R} - I_s e^{V_1/U_T}$$

$$C\frac{dV_{out}}{dt} = I_s e^{(V_2 - V_{out})/U_T} - \frac{V_{out}}{R}$$

$$V_1 = \frac{1}{2}(V_{in} + V_{out})$$

Steady-State $\frac{dV_2}{dt} = \frac{dV_{out}}{dt} = 0$

$$I_s e^{(V_2 - V_{out})/U_T} = \frac{V_{out}}{R} \qquad \frac{V_{dd} - V_2}{R} = I_s e^{V_1/U_T}$$

Expand around the solution ($V_{out,0}$, $V_{2,0}$, etc.)

$$V_1 = V_{1,0} + \Delta V_1 \qquad V_{out} = V_{out,0} + \Delta V_{out,0}$$

$$V_2 = V_{2,0} + \Delta V_2 \qquad V_{in} = V_{in,0} + \Delta V_{in,0}$$

$$\frac{RC}{V_{dd} - V_{2,0}}\frac{dV_2}{dt} = 1 - \frac{\Delta V_2}{V_{dd} - V_{2,0}} - e^{\Delta V_1/U_T}$$

$$\frac{RC}{V_{dd} - V_{out,0}}\frac{dV_{out}}{dt} = e^{(\Delta V_2 - \Delta V_{out})/U_T} - 1 - \frac{\Delta V_{out}}{V_{out,0}}$$

$$\frac{V_{dd} - V_{2,0} - \Delta V_2}{R} \to \frac{V_{dd} - V_{2,0}}{R}\left(1 - \frac{\Delta V_2}{V_{dd} - V_{2,0}}\right)$$

Linearize: $e^x \approx 1 + x + O(2)$

$$\frac{RC}{V_{dd} - V_{2,0}}\frac{dV_2}{dt} = -\frac{\Delta V_2}{V_{dd} - V_{2,0}} - \frac{\Delta V_1}{U_T}$$

$$\frac{RC}{V_{dd} - V_{out,0}}\frac{dV_{out}}{dt} = \frac{\Delta V_2 - \Delta V_{out}}{U_T} - \frac{\Delta V_{out}}{V_{out,0}}$$

Normalize:

$$u = \frac{\Delta V_{in}}{U_T} \quad A_v = \frac{V_{dd} - V_{2,0}}{2U_T}$$

$$x_1 = \frac{\Delta V_2}{U_T} \quad x_2 = \frac{\Delta V_{out}}{U_T}$$

$$\tau_1 = RC \quad \tau_2 = RC\frac{U_T}{V_{dd} - V_{out,0}}$$

$$\tau_1 \frac{dx_1}{dt} + x_1 = -A_v(u + x_2)$$

$$\tau_2 \frac{dx_2}{dt} + x_2 = x_1$$

Continuous-Time Linear State

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(\mathbf{t})\mathbf{u}(t)$$

General Multiple Input, Multiple Output (MIMO) Case

Time-Invariant (LTI) Form:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

$$\xrightarrow[\mathbf{x}_1 = \mathbf{T}\mathbf{x}]{\text{For}}$$

$$\bar{\mathbf{A}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1} \qquad \bar{\mathbf{B}} = \mathbf{T}\mathbf{B} \qquad \bar{\mathbf{C}} = \mathbf{C}\mathbf{T}^{-1}$$

$$\frac{d\mathbf{x}(t)}{dt} = \bar{\mathbf{A}}\mathbf{x}(t) + \bar{\mathbf{B}}\mathbf{u}(t)$$

$$\mathbf{y}(t) = \bar{\mathbf{C}}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

Two *algebrically equivalent* systems

State variables change,
input and output are the same

Single Input, Single Output (SISO) Case

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{b}(t)u(t)$$

$$y(t) = \mathbf{c}(t)^T\mathbf{x}(t) + du(t)$$

A(t): Defining Transfer Function States
(matrix, nxn matrix for n states)
→ a rational polynomial function in s

Continuous-Time (CT) Linear State
→ Discrete-Time (DT) Linear State

Classic CT form

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(\mathbf{t})\mathbf{u}(t)$$

To move to sampled variables:

$\mathbf{x}$[n], $t = nT_0$

$$\frac{d\mathbf{x}(t)}{dt} \approx \frac{\mathbf{x}(t+\Delta) - \mathbf{x}(t)}{\Delta}$$

Approximate derivative, time sample spacing (Δ)

With the modified equations

$$\mathbf{x}(t+\Delta) - \mathbf{x}(t) \approx \Delta\left(\mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)\right)$$

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(\mathbf{t})\mathbf{u}(t)$$

$$\mathbf{x}[n+1] - \mathbf{x}[n] \approx \Delta\left(\mathbf{A}[n]\mathbf{x}[n] + \mathbf{B}[n]\mathbf{u}[n]\right)$$

$$\mathbf{y}[n] = \mathbf{C}[n]\mathbf{x}[n] + \mathbf{D}[\mathbf{n}]\mathbf{u}[n]$$

In this formulation, one expects roughly similar behavior to the CT case for small Δ

And yet, some books and resources use

$$\mathbf{x}(t+\Delta) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(\mathbf{t})\mathbf{u}(t)$$

And properly written

$$\mathbf{x}[n+1] = \mathbf{A}[n]\mathbf{x}[n] + \mathbf{B}[n]\mathbf{u}[n]$$

$$\mathbf{y}[n] = \mathbf{C}[n]\mathbf{x}[n] + \mathbf{D}(\mathbf{t})\mathbf{u}[n]$$

Stability and Other Properties are Different

CT Form: eigenvalues of A < 0

Laplace Transform

DT Form: | eigenvalues of A | < 1

z-Transform

State Variable Dynamics:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}(\mathbf{t})\mathbf{x}(t)$$

For constant **A** (LTI):

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t) \longrightarrow \quad \mathbf{x} = \mathbf{E}\mathbf{x_1}$$

$$\mathbf{A} = \mathbf{E}\mathbf{\Lambda}\mathbf{E}^{-1}$$

E = eigenvectors of A

$$\Lambda = \mathrm{diag}(\lambda_1 ... \lambda_n)$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$$

↓ Eigenvector basis

$$\frac{d\mathbf{x_1}(t)}{dt} = \mathbf{\Lambda}\mathbf{x_1}(t)$$

$$x_{1,k}(t) = x_{1,k}(0)e^{\lambda_\mathbf{k} t}$$

Solution for variable A(t)? Peano-Baker Series

$$\mathbf{\Phi}(t, t_0) = \mathbf{I} + \int_{t_0}^{t} \mathbf{A}(s_1)ds_1 + \int_{t_0}^{t} \mathbf{A}(s_1) \int_{t_0}^{s_1} \mathbf{A}(s_2)ds_2 ds_1 + \int_{t_0}^{t} \mathbf{A}(s_1) \int_{t_0}^{s_1} \mathbf{A}(s_2) \int_{t_0}^{s_2} \mathbf{A}(s_3)ds_3 ds_2 ds_1 + ...$$

$$\frac{d\mathbf{\Phi}(t, t_0)}{dt} = \mathbf{A}(t)\mathbf{\Phi}(t, t_0)$$

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\mathbf{x}(0)$$

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_1)\mathbf{\Phi}(t_1, t_0)\mathbf{x}(0)$$ → Solution depends on initial conditions

Linear System Solution:

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\mathbf{x}(0) + \int_{t_0}^{t} \mathbf{\Phi}(t, t_k)\mathbf{B}(t_k)\mathbf{u}(t_k)dt_k$$

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{\Phi}(t, t_0)\mathbf{x}(0) + \int_{t_0}^{t} \mathbf{C}(t)\mathbf{\Phi}(t, t_k)\mathbf{B}(t_k)\mathbf{u}(t_k)dt_k + \mathbf{D}(t)\mathbf{u}(t_k)$$

For constant **A** (LTI):

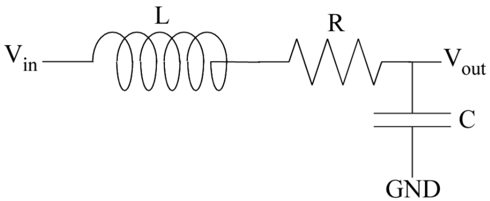
$$\mathbf{\Phi}(t, t_0) = \mathbf{I} + \int_{t_0}^{t} \mathbf{A}ds_1 + \int_{t_0}^{t} \mathbf{A} \int_{t_0}^{s_1} \mathbf{A}ds_2 ds_1 + \int_{t_0}^{t} \mathbf{A} \int_{t_0}^{s_1} \mathbf{A} \int_{t_0}^{s_2} \mathbf{A}ds_3 ds_2 ds_1 + ...$$

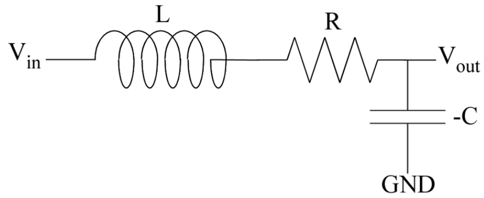
$$= \sum_{k=0}^{\infty} \frac{(t - t_0)^k}{k!}\mathbf{A}^k = e^{\mathbf{A}(t - t_0)}$$

$$\mathbf{x}(t) = e^{\mathbf{A}(t - t_0)}\mathbf{x}(0) + \int_{t_0}^{t} e^{\mathbf{A}(t - t_1)}\mathbf{B}\mathbf{u}(t_1)dt_1$$

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}(t - t_0)}\mathbf{x}(0) + \int_{t_0}^{t} \mathbf{C}e^{\mathbf{A}(t - t_1)}\mathbf{B}\mathbf{u}(t_1)dt_1 + \mathbf{D}(t)\mathbf{u}(t)$$

Control Example with Linear Circuits





$$\frac{V_{out}}{V_{in}} = \frac{1}{1 + sRC + s^2 LC} \qquad \frac{V_{out}}{V_{in}} = \frac{1}{1 - sRC - s^2 LC}$$

$$\tau^2 = LC \quad \tau/Q = RC, Q = \frac{1}{R}\sqrt{\frac{L}{C}}$$

$$\tau^2 \frac{d^2 V_{out}}{dt^2} + \frac{\tau}{Q}\frac{dV_{out}}{dt} + V_{out} = V_{in} \qquad \tau^2 \frac{d^2 V_{out}}{dt^2} + \frac{\tau}{Q}\frac{dV_{out}}{dt} - V_{out} = -V_{in}$$

$$t = t_1 \tau$$

$$\frac{d^2 V_{out}}{dt_1^2} + \frac{1}{Q}\frac{dV_{out}}{dt_1} + V_{out} = V_{in} \qquad \frac{d^2 V_{out}}{dt_1^2} + \frac{1}{Q}\frac{dV_{out}}{dt_1} - V_{out} = -V_{in}$$

→ Same equations as one degree of freedom inverted pendulum (stable and unstable steady state) (sign of Torque?)

$$\mathbf{x}(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} V_{out} \\ \frac{dV_{out}}{dt_1} \end{pmatrix} \longrightarrow \frac{dx_1}{dt_1} = x_2 \qquad \mathbf{u}(t) = \begin{pmatrix} 0 \\ V_{in}(t) \end{pmatrix}$$

$$\frac{d\mathbf{x}}{dt_1} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(\mathbf{t})$$

Stable (Exponentially)

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & -1/Q \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Eigenvalues: $\lambda^2 + \frac{1}{Q}\lambda + 1 = 0$

$$\lambda = -\frac{1}{2Q}\left(1 \pm \sqrt{1 - 4Q^2}\right)$$

Two stable roots (real or complex)
Re $\lambda < 0$

Unstable

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & -1/Q \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

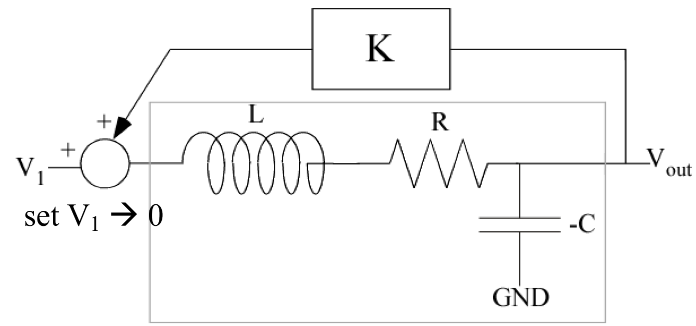
Eigenvalues: $\lambda^2 + \frac{1}{Q}\lambda - 1 = 0$

$$\lambda = -\frac{1}{2Q}\left(1 \pm \sqrt{1 + 4Q^2}\right)$$

Two real roots
1. Re $\lambda < 0$
2. Re $\lambda > 0$

Saddle Node Dynamics

Control around unstable node



$u_2 = K x_1 \longrightarrow$ Stable for any K ?

$$\frac{d\mathbf{x}}{dt_1} = \begin{pmatrix} 0 & 1 \\ 1 & -1/Q \end{pmatrix}\mathbf{x} + \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ K x_1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 - K & -1/Q \end{pmatrix}\mathbf{x}$$

$$\lambda^2 + \frac{1}{Q}\lambda + K - 1 = 0$$

$$\lambda = -\frac{1}{2Q}\left(1 \pm \sqrt{1 - 4(K-1)Q^2}\right)$$

$K = 1, \lambda = 0, -1/Q$ Stable (marginally)

$K > 1$, Re $\lambda_1, \lambda_2 < 0$ Stable (exponential)

Types of Stability

General Linear State Equations

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(\mathbf{t})\mathbf{u}(t)$$

Unstable $||y(t)||$ Unbounded somewhere $0 < t < \infty$

LTI: One $\lambda_1, \lambda_2, ..., \lambda_n > 0$

Marginally Stable $||y(t)||$ Always bounded $0 < t < \infty$

LTI: All $\lambda_1, \lambda_2, ..., \lambda_n \leq 0$

Asymptotically stable $||y(t)|| \to 0$ as $t \to \infty$

LTI: All $\lambda_1, \lambda_2, ..., \lambda_n < 0$

Exponentially stable $||y(t)|| < Ce^{\lambda t}$ positive C, negative λ, as $t \to \infty$

LTI: All $\lambda_1, \lambda_2, ..., \lambda_n < 0$

Bounded-Input, Bounded-Output (BIBO) Stability:

stability extended to include all bounded inputs

Stability for Different State Matrices

LTI:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

Stability → Eigenvalues of **A**

$$\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \qquad \lambda^2 + \lambda - 1 = 0 \qquad \lambda = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}$$

Unstable, one positive e-value

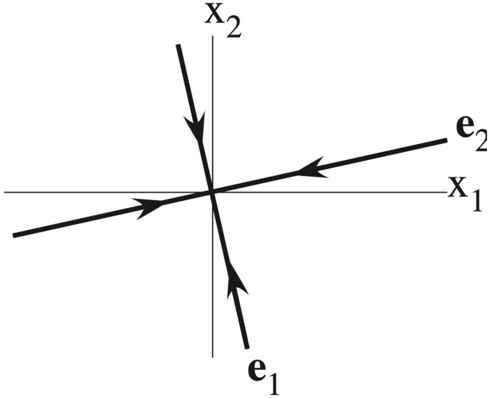
$$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \qquad \lambda^2 + \lambda + 1 = 0 \qquad \lambda = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$$

Stable, complex e-values

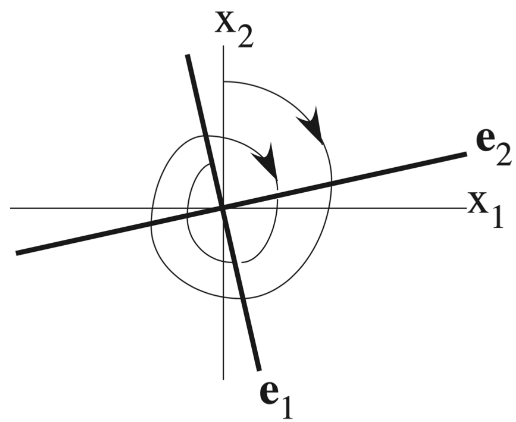
Local Two-State Stability

Stable equilibriums

λ_1, λ_2 Real and negative

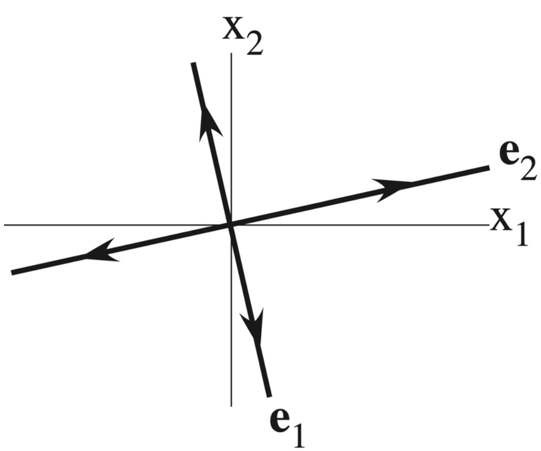


λ_1, λ_2 Complex, $Re(\lambda_1, \lambda_2) < 0$

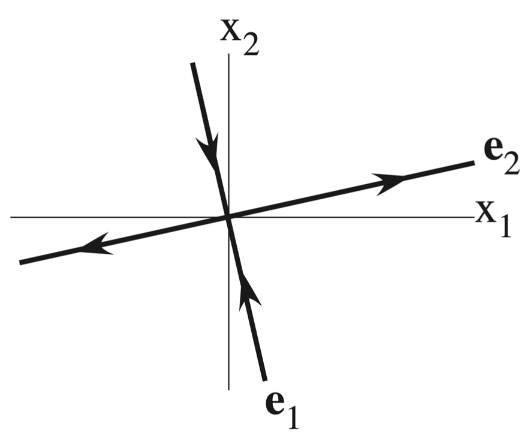


Unstable equilibriums

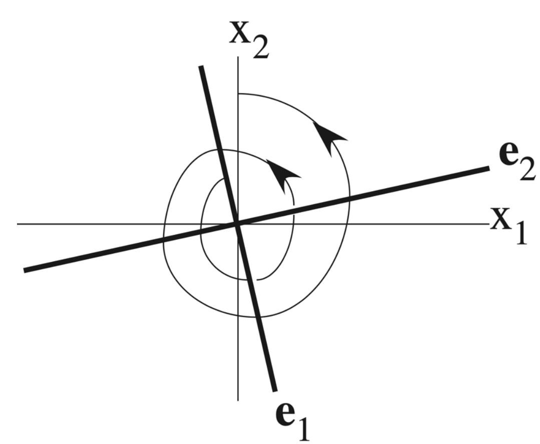
λ_1, λ_2 Real and positive



λ_1, λ_2 Real, $\lambda_1 > 0$, $\lambda_2 < 0$

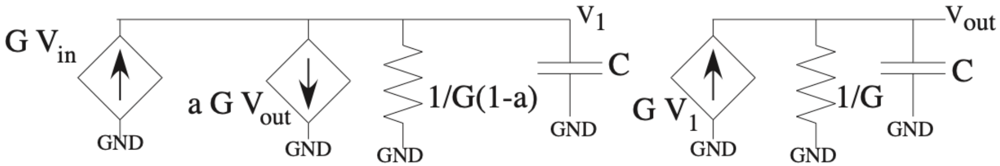


λ_1, λ_2 Complex, $Re(\lambda_1, \lambda_2) > 0$



A Circuit to Control

Build state equation Matrix for the circuit



$$C\frac{dV_1}{dt} = GV_{in} - G(1-a)V_1 - aGV_{out} \qquad C\frac{dV_{out}}{dt} = GV_1 - GV_{out}$$

Define Time Constant $\tau = \frac{C}{G} \longrightarrow t = \tau t_1$

$$\frac{dV_1}{dt_1} = V_{in} - (1-a)V_1 - aV_{out} \qquad \frac{dV_{out}}{dt_1} = V_1 - V_{out}$$

Define: $\mathbf{x}(t) = \begin{pmatrix} V_1(t) \\ V_{out}(t) \end{pmatrix} \quad \mathbf{u}(t) = \begin{pmatrix} V_{in}(t) \\ 0 \end{pmatrix}$

$$\frac{d\mathbf{x}(t)}{dt_1} = \begin{pmatrix} -(1-a) & -a \\ 1 & -1 \end{pmatrix} \mathbf{x}(t) + \mathbf{Iu}(t)$$

Eigenvalues for **A**:

$$(\lambda + 1 - a)(\lambda + 1) + a = 0$$

$$\lambda^2 + (2-a)\lambda + 1 = 0$$

$$\lambda = -(1 - \frac{a}{2}) \pm \sqrt{\left(1 - \frac{a}{2}\right)^2 - 1}$$

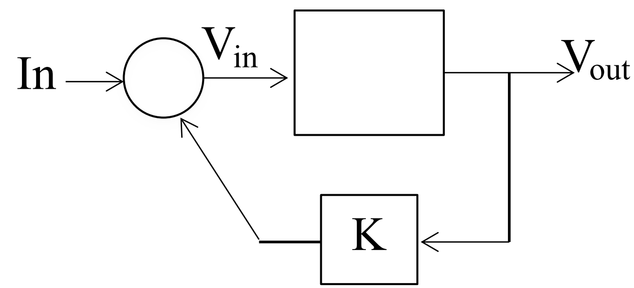
$$a = 1, \lambda = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$$

(stable, complex)

$$a = 3, \lambda = \frac{1}{2} \pm j\frac{\sqrt{3}}{2}$$

(unstable, complex)

Can we control this system for a=3?



$$\frac{d\mathbf{x}(t)}{dt_1} = \begin{pmatrix} -(1-a) & -a \\ 1 & -1 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} KV_{out}(t) \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -(1-a) & K-a \\ 1 & -1 \end{pmatrix} \mathbf{x}(t)$$

$$\lambda^2 + (2-a)\lambda + 1 - K = 0$$

$$\lambda = -(1 - \frac{a}{2}) \pm \sqrt{\left(1 - \frac{a}{2}\right)^2 - 1 + K}$$

At least one roots remains positive at a=3.