CHAPTER

11

SECOND-ORDER SECTION

We have done integration and differentiation with simple, single–time-constant circuits that had $\tau s + 1$ in the denominator of their transfer functions. These systems gave an exponentially damped response to step or impulse inputs. In Chapter 8, we showed how a second-order system can give rise to a sinusoidal response. In this chapter, we will discuss a simple circuit that can generate a sinusoidal response. We call this circuit the **second-order section**; we can use it to generate any response that can be represented by two poles in the complex plane, where the two poles have both real and imaginary parts. With this circuit, we can adjust the positions of the complex-conjugate poles anywhere in the plane.

The second-order circuit is shown in Figure 11.1; it contains two cascaded follower–integrator circuits and an extra amplifier. The capacitance C is the same for both stages ($C_1 = C_2 = C$), and the transconductance of the two feed-forward amplifiers, A1 and A2, are the same: $G_1 = G_2 = G$ (approximately—if G is defined as the average of G_1 and G_2, small differences will have no first-order effect on the parameters of the response). We obtain an oscillatory response by adding the *feedback amplifier* A3. This amplifier has transconductance G_3, and its output current is proportional to the difference between V_2 and V_3, but the sign of the feedback is *positive*; for small signals, I_3 is equal to $G_3(V_2 - V_3)$.

If we reduce the feedback to zero by shutting off the bias current in A3, each follower–integrator circuit will have the transfer function given

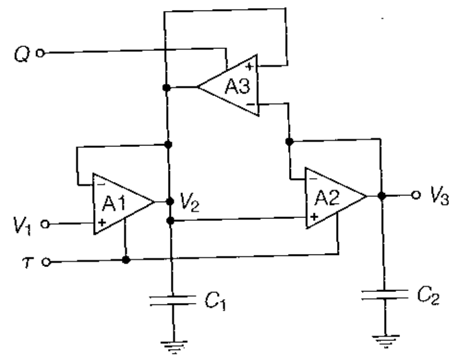

FIGURE 11.1 Circuit diagram of the second-order section. The amplifier A3 tends to keep V_2 ahead of V_1, once V_2 has gained the lead. For that reason, A3 has a destabilizing effect on the circuit behavior.

in Equation 9.3 (p. 148). Two follower–integrator circuits in cascade give an overall transfer function that is the product of the individual transfer functions, so there are two poles at $s = -1/\tau$:

$$\frac{V_3}{V_1} = \left(\frac{1}{\tau s + 1}\right)^2 = \frac{1}{\tau^2 s^2 + 2\tau s + 1}$$

We can understand the contribution of A3 to the response by following through the dynamics of the system when a perturbation is applied to the input. Suppose we begin with the input biased to some quiescent voltage level. In the steady state, all three voltages will settle down, and V_2 and V_3 will both be equal to V_1. If we apply to V_1 a small step function on top of this DC level, V_2 starts increasing, because we are charging up the first capacitor C1. Eventually, V_2 gets a little ahead of V_3, and then amplifier A3 makes V_2 increase even faster. Once V_2 is increasing, the action of A3 is to keep it increasing; the feedback around the loop is positive. If we set the transconductance G_3 of amplifier A3 high enough, V_2 will increase too fast, and the circuit will become unstable.

SMALL-SIGNAL ANALYSIS

A2 is a follower–integrator circuit, so, from Equation 9.3 (p. 148),

$$V_3 = \frac{V_2}{\tau s + 1} \tag{11.1}$$

where $\tau = C/G_2$. The current I_1 coming out of amplifier A1 is proportional to the difference between V_1 and V_2, its two inputs:

$$I_1 = G_1(V_1 - V_2) \tag{11.2}$$

We can describe I_3, the output of A3, in the same way:

$$I_3 = G_3(V_2 - V_3) \tag{11.3}$$

We need an equation for V_3 in terms of V_1. Combining the two currents into the capacitor (Equations 11.2 and 11.3), we obtain

$$C\frac{dV_2}{dt} = G_1(V_1 - V_2) + G_3(V_2 - V_3)$$

Using $s = d/dt$ and collecting terms,

$$V_2(sC + G_1 - G_3) = G_1 V_1 - G_3 V_3$$

Substituting V_2 from Equation 11.1, and simplifying using $\tau = C/G_1 = C/G_2$ and $\alpha = G_3/(G_1 + G_2)$,

$$H(s) = \frac{V_3}{V_1} = \frac{1}{\tau^2 s^2 + 2\tau s(1-\alpha) + 1} \tag{11.4}$$

Equation 11.4 is the transfer function for the circuit; as we expected, it is a second-order expression in τs. The parameter α is the ratio of the feedback transconductance G_3 to the total forward transconductance $G_1 + G_2$. If α is equal to 0, Equation 11.4 should give the response of two first-order sections. The denominator is $(\tau s + 1)^2$, just as we expected.

We also can see that, when α is equal to 1, the center term in the denominator becomes 0, and we get

$$\frac{V_3}{V_1} = \frac{1}{\tau^2 s^2 + 1}$$

Under these conditions, the roots of the natural system response are thus

$$\tau^2 s^2 = -1 \qquad \text{or} \qquad \tau s = \pm j$$

Complex Roots

We can put the poles on the imaginary axis when α is equal to 1, or right down on the real axis when α is equal to 0. Now our job is to determine

1. Where the poles are located when α is neither 0 nor 1
2. How the system responds under such conditions

We can write the transfer function

$$\frac{V_3}{V_1} = \frac{1}{(\tau s - \tau R_1)(\tau s - \tau R_2)} \tag{11.5}$$

where R_1 and R_2 are the roots of the denominator in the s-plane. We can define the position of any root as $Re^{j\theta}$, where R is the distance to the root from the origin and θ is the angle from the positive real axis to the root; that is just the polar form of a complex number. As we discussed in Chapter 8, complex roots of real polynomials must occur as complex-conjugate pairs:

$$R_1 = Re^{j\theta} \qquad \text{and} \qquad R_2 = Re^{-j\theta} \tag{11.6}$$

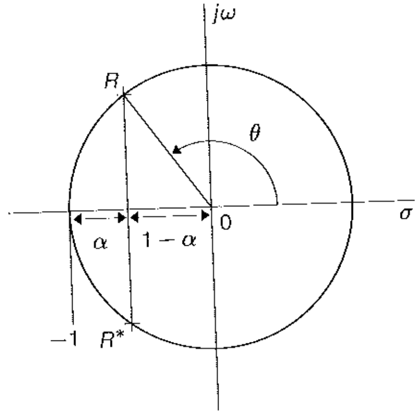

FIGURE 11.2 Complex-plane representation of small-signal behavior of the second-order section. The effect of $\alpha = G_3/(G_1 + G_2)$ is to move the poles toward the $j\omega$ axis. For α greater than 1, the circuit is unstable.

We will solve for R and θ in terms of α and τ by comparing the denominator of Equation 11.5 with that of Equation 11.4. Substituting Equation 11.6 into Equation 11.5, we obtain

$$\tau^2 s^2 + 2\tau s(1-\alpha) + 1 = \tau^2 s^2 - \tau^2 sR\left(e^{j\theta} + e^{-j\theta}\right) + \tau^2 R^2$$

Using the identity $2\cos\theta = e^{j\theta} + e^{-j\theta}$, we arrive at an important result:

$$\tau = \frac{1}{R} \qquad \text{and} \qquad -\cos\theta = 1 - \alpha$$

The roots are located on a circle of radius $1/\tau$; they move to the right from the negative real axis as we increase the transconductance of the feedback amplifier. The angle of the roots is determined by the ratio of feedback transconductance to forward transconductance, and is independent of the absolute value of τ.

We can normalize the plot, so the roots lie on the unit circle, by expressing all distances in the plane in units of $1/\tau$, as shown in Figure 11.2. From this construction, we have the following important relation between the major circuit variables:

$$(\omega\tau)^2 + (\sigma\tau)^2 = 1 \tag{11.7}$$

We can see from Figure 11.2 that $\cos\theta$ is the projection of either root onto the σ axis. The poles are $1-\alpha$ to the left of the origin. The real part σ of both roots is given by

$$-\sigma\tau = 1 - \alpha \tag{11.8}$$

When α is 1, the real part is 0, and we are left with a pair of roots on the $\pm j\omega$ axis. When α is 0, the root pair is at the point -1 on the negative real axis; for α greater than 0, the distance between that point and the real part of the roots is just α.

We now have a way to visualize the effect of the feedback ratio α on the location of the roots: α is the horizontal distance from the roots to their original position when there was no feedback in the circuit. The roots start at -1 on the real axis; as we increase G_3, we push them to the right, decreasing the magnitude of the damping constant σ. Eventually, we push them across the $j\omega$ axis and the circuit begins to oscillate. We have shown that we have an independent control on the location of the roots on the circle. The radius of the circle is determined by G. By changing α to be equal to $G_3/(2G)$, we can locate the roots anywhere on the circle. As we change the location of the roots, we change the response of the circuit. That is why the second-order section is such a useful device.

Second-order systems often are characterized in terms of a Q parameter, defined by $Q = -1/(2\sigma\tau)$, or by the transfer function expression

$$H(s) = \frac{1}{\tau^2 s^2 + \frac{1}{Q}\tau s + 1}$$

By comparing this expression with Equation 11.4 or Equation 11.8, we find that

$$Q = \frac{1}{2(1-\alpha)}$$

Note that Q starts from 0.5 with no feedback ($\alpha = 0$), and grows without bound as the feedback gain approaches the total forward gain ($\alpha = 1$ or $G_3 = G_1 + G_2$); beyond this point, small signals grow exponentially—the circuit is unstable.

Transient Response

In Chapter 8, we saw that the natural response of a linear system always could be written

$$V(t) = e^{st} = e^{\sigma t}e^{j\omega t}$$

where the value of s is given by the root or conjugate pair of roots of the denominator of the transfer function. The impulse response of the circuit, for positive t, is of the same form as this natural response.

Depending on the value of α, the behavior of the second-order section may be best described in either the time domain or the frequency domain. The impulse response of the circuit when the roots are on the real axis is just an exponential; the response does not oscillate at all, and there is therefore no frequency associated with it. It is simply a dying exponential. When the roots are on the imaginary axis, the circuit is an oscillator—it just sits there and oscillates, on and on and on. For α between 0 and 1, the impulse response is a damped sine wave, as shown in Figure 11.3, because s has both a real and an imaginary component. The $e^{j\omega t}$ is an oscillating response, composed of sines and cosines. The $e^{\sigma t}$ is the damping term, as long as σ is negative.

The values of ω and σ can be determined directly from Figure 11.3. The duration of one cycle is 460 microseconds. A cycle is 2π radians; ω is thus equal to 1.37×10^4 radians per second, which is about $1/\tau$. The wave damps by a factor of $1/e$ in 2.6 milliseconds. The damping constant σ is thus -3.85×10^2 per second.

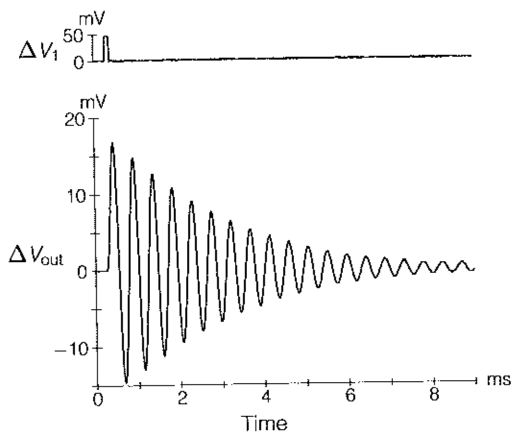

FIGURE 11.3 Measured impulse response of the second-order section for $\alpha \approx 0.97$.

The value of α from Equation 11.8 is

$$\alpha = 1 + \sigma\tau = 0.97$$

The circuit becomes unstable when G_3 is greater than $2G$. We can think about the onset of instability in the following way. There are two amplifiers (A1 and A2) with negative feedback, but there is only one amplifier (A3) with positive feedback. Negative feedback damps the response, and positive feedback reduces the damping. To make the circuit unstable, A3 must provide as much current as do the two amplifiers, A1 and A2, that provide the damping. When the two effects are equal, the circuit is just marginally damped. As we increase G_3 above $2G$, the damping becomes negative, and the response becomes an exponentially *growing* sine wave. Exponential growth is an explosive kind of thing, so the second-order section rapidly leaves the small-signal regime, and becomes dominated by large-signal effects, as we discuss later in this chapter.

Frequency Response

When the damping of the circuit is low, we find it natural to view the response as a function of frequency. The frequency response of the second-order section of Figure 11.1 is shown in Figure 11.4 for a number of values of α. The highest peak corresponds to the setting used for the transient response shown in Figure 11.3.

We can evaluate the frequency response by substituting $j\omega$ for s into Equation 11.4:

$$\frac{V_3}{V_1} = \frac{1}{-\omega^2\tau^2 + j2\omega\tau(1-\alpha) + 1}$$

We can simplify the algebra by computing D, the magnitude of the denominator of the transfer function, in terms of a normalized frequency $f = \omega\tau$ and

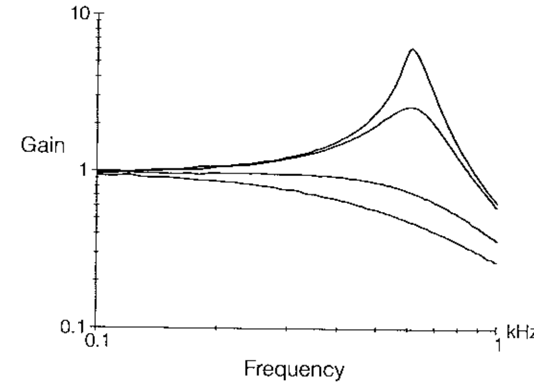

FIGURE 11.4 Measured frequency response of the second-order section for several values of α. The highest curve is for $\alpha \approx 0.94$, slightly lower than the value used for the transient behavior shown in Figure 11.3. The lowest curve is for $\alpha = 0$, and the second curve is very near $Q^2 = 1/2$.

$2(1-\alpha) = 1/Q$. Using the fact that the magnitude of a complex number can be computed from the Pythagorean theorem, we have

$$D^2 = \left(1 - f^2\right)^2 + \frac{f^2}{Q^2} = f^4 - f^2\left(2 - \frac{1}{Q^2}\right) + 1 \tag{11.9}$$

It is convenient to plot the log of the magnitude of the transfer function as a function of $\log f$. But

$$\log\left|\frac{V_3}{V_1}\right| = -\tfrac{1}{2}\log(D^2)$$

Hence, we can reason about the response directly from the behavior of D^2. When f is small, the f^4 term is much smaller than is the f^2 term, so

$$\log\left|\frac{V_3}{V_1}\right| = -\tfrac{1}{2}\log\left(1 - f^2\left(2 - \frac{1}{Q^2}\right)\right) \approx f^2\left(1 - \frac{1}{2Q^2}\right)$$

At low frequencies (f much less than 1), the response grows larger as f is increased, provided Q^2 is greater than $1/2$ or Q is greater than 0.707. At some frequency, the f^4 term is no longer negligible, and it starts canceling the effect of the f^2 term. Above that frequency, f^4 increases much faster than f^2 does, so the response decreases. Eventually, the f^4 term is much larger than the other terms are, so the response decreases as $1/f^2$, because

$$-\tfrac{1}{2}\log\left(f^4\right) = -2\log f$$

The plot decreases with a slope of -2 when f is much greater than 1.

So we know the asymptotes. Near zero frequency, the gain is 1 and the response is flat (independent of frequency); at very high frequencies, the slope on a log scale approaches -2, because the response is proportional to $1/f^2$; between the two extremes, there is a maximum. The f^2 term increases before the f^4 term does; as f^2 increases, so does the response. Eventually, the f^4 term becomes large enough to dominate the f^2 term, and then the response begins to decrease.

The response will be a maximum where D^2 is a minimum—that is, where the derivative is zero:

$$\frac{dD^2}{df} = 4f^3 - 2f\left(2 - \frac{1}{Q^2}\right) = 0$$

$$f_{\max}^2 = 1 - \frac{1}{2Q^2} \tag{11.10}$$

Equation 11.10 tells us *where* the peak in the response curve is. Now we can take that maximum frequency from Equation 11.10 and put it back into the transfer function Equation 11.9 to find the *value* of the denominator at the peak:

$$D_{\max}^2 = \frac{1}{Q^2}\left(1 - \frac{1}{4Q^2}\right) \tag{11.11}$$

Equation 11.11 gives a maximum value of the transfer function

$$\left.\frac{V_3}{V_1}\right|_{\max} = \frac{Q}{\sqrt{1 - \frac{1}{4Q^2}}}$$

So, as Q becomes large, the height of the peak approaches Q, and the peak frequency approaches $1/\tau$.

When Q^2 is equal to $1/2$, the peak gain is 1 at zero frequency and the response is maximally flat (that is, the lowest-order frequency dependence is f^4). For lower Q values, the gain drops off quadratically with frequency.

LARGE-SIGNAL BEHAVIOR

Thus far, we have been concerned with the second-order section as a linear system. The linear approximation is valid for small amplitudes of oscillation. As we might expect, the circuit has all the slew-rate limitations we saw for first-order filters. When the second-order section becomes slew-rate limited, however, its behavior is much more exciting than that of its first-order cousins. When the circuit that generated the small-signal impulse response of Figure 11.3 is subjected to a large impulse input, it breaks into a sustained **limit-cycle oscillation**, as shown in Figure 11.5. The amplitude of the oscillation is the full range of the power supply. Thus, the circuit that is perfectly stable for small signals becomes wildly unstable for large signals. We need a little imagination to visualize a system controlled by such a circuit, gripped by recurring seizures of this violent electronic epilepsy. As with any pathology, we must understand the etiology, and take precautions against any possible onset of the disease.

We can analyze this grotesque behavior by realizing that the input voltages to all three amplifiers are many $kT/(q\kappa)$ units apart over almost all parts of the waveform. Under these conditions, the currents out of the amplifiers are constant,

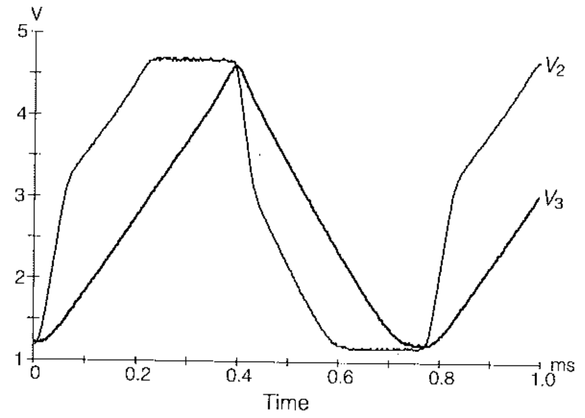

FIGURE 11.5 Limit-cycle oscillation of the second-order section for $\alpha \approx 0.97$, the same setting under which the small-signal response of Figure 11.3 was obtained. The output traverses the entire range from ground to V_{DD}.

equal to plus or minus the value of the bias current in the differential-pair current source. This current was called I_b in Chapter 5. We designate the bias currents in amplifiers A1, A2, and A3 as I_1, I_2, and I_3, respectively. The small-signal transconductances of the amplifiers are directly proportional to these currents. We should thus be able to relate the large-signal instability to the small-signal properties of the second-order section.

Recall what the feedback amplifier A3 is doing: It is sensing the difference between V_2 and V_3 and feeding a current back into V_2 to increase dV_2/dt. In the slew-rate limit, V_3 does not have the benefit of positive feedback, and thus it is lagging behind V_2. As long as V_2 is less than V_1, both A1 and A3 are feeding their maximum current into C_1. The steep ascent of V_2 thus has the slope

$$\frac{dV_2}{dt} = \frac{I_3 + I_1}{C_1} \qquad \text{where} \qquad V_3 \ll V_2 \ll V_1$$

Once V_2 passes through V_1, the current out of A1 changes sign, and

$$\frac{dV_2}{dt} = \frac{I_3 - I_1}{C_1} \qquad \text{where} \qquad V_3 \ll V_2 \gg V_1$$

For the entire first half of the oscillation cycle, V_2 is well ahead of V_3, and the output current from A2 is just I_2:

$$\frac{dV_3}{dt} = \frac{I_2}{C_2} \qquad \text{where} \qquad V_3 \ll V_2$$

Stability Limit

The large-signal oscillation is sustained if V_2 reaches the rail before V_3 catches up to V_2. As the feedback current I_3 is decreased, there is a point at which V_2 and V_3 reach the rail at virtually the same time. The oscillation waveform under

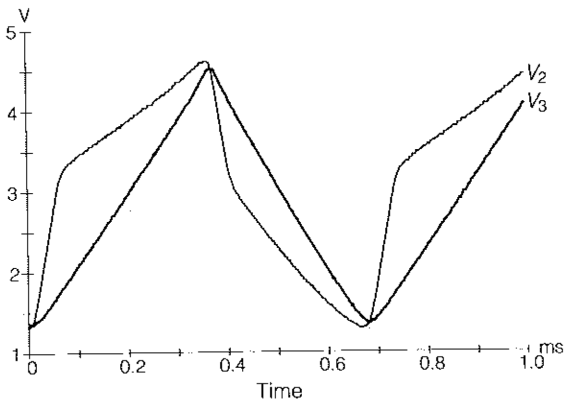

FIGURE 11.6 Minimum limit-cycle oscillation of the second-order section for $\alpha \approx 0.81$. Lower settings of α lead to a damped response for all amplitudes of input signals.

those conditions is shown in Figure 11.6. If I_3 is decreased below the value shown in the figure, oscillation ceases. We can solve for the value of I_3 that will just sustain a large-signal limit cycle by equating the time required for V_2 to reach V_{DD} with that required for V_3 to do so.

The time t_0 required for the output voltage to ramp from zero to V_{DD} is the total voltage excursion divided by the rate at which the voltage increases:

$$t_0 = \frac{C_2 V_{DD}}{I_2}$$

Similarly, the time t_l for V_2 to reach V_1 is

$$t_l = \frac{C_1 V_1}{I_3 + I_1}$$

and the time t_h for V_2 to ramp from V_1 to V_{DD} is

$$t_h = \frac{C_1(V_{DD} - V_1)}{I_3 - I_1}$$

Marginal oscillation occurs when

$$t_l + t_h = t_0 \tag{11.12}$$

For the example shown in Figure 11.6, V_1 is approximately $V_{DD}/2$, and the result is independent of the value of V_{DD}. We assume that $I_1 = I_2 = I$ and that $C_1 = C_2$, and we divide all terms by I to obtain a dimensionless form of Equation 11.12:

$$\frac{1}{2\alpha+1} + \frac{1}{2\alpha-1} = 2 \tag{11.13}$$

where, as before, α is equal to $I_3/(2I)$.

Equation 11.13 can be simplified to

$$4\alpha^2 - 2\alpha - 1 = 0 \tag{11.14}$$

The solution of Equation 11.14 is

$$\alpha = \frac{1+\sqrt{5}}{4} = 0.809$$

The astute observer will immediately recognize that this number is one-half of the **golden ratio** of classical Greek antiquity.

What a quaint result! We cannot allow current in the feedback amplifier to exceed 0.809 times the sum of the currents in the forward amplifiers if we wish the circuit to be stable for all possible inputs. In the realm of small signals, we could increase that ratio to nearly 1. But small-signal linear behavior is not limiting us here. We are strictly at the mercy of the nonlinearities.

Small-Signal Behavior at Stability Limit

We have found the maximum value of α at which it is safe to operate the second-order section. If we adjust the current in amplifier A3 such that large-signal oscillations are just marginally stable, we can examine how the section will operate in its small-signal regime. The small-signal response to a step input with $\alpha = 0.809$ is shown in Figure 11.7. From Equation 11.8 we can determine that $\sigma\tau = 0.191$ is the damping constant expected under these conditions.

The natural period of the oscillation is $2\pi/\omega$. From Equation 11.7, we find that $\omega\tau$ is equal to 0.9816 for the present conditions. In one cycle of oscillation, we expect the waveform to have been damped by

$$\frac{V(t=0)}{V(t=2\pi/\omega)} = e^{2\pi\sigma/\omega} \tag{11.15}$$

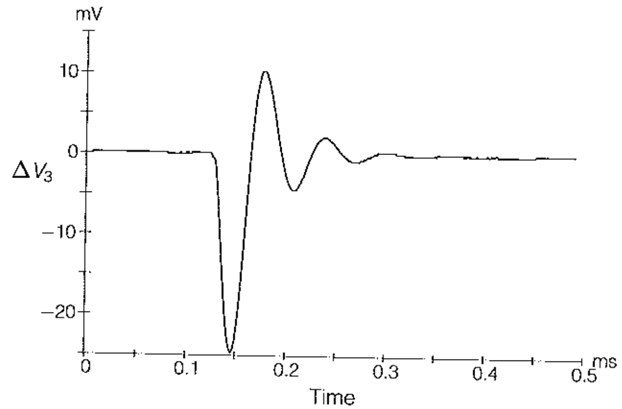

FIGURE 11.7 Small-signal response of the second-order section measured for $\alpha \approx 0.81$. This setting gave the minimum large-signal limit-cycle shown in Figure 11.6.

For operation at the large-signal stability limit, Equation 11.15 predicts a factor of 3.4 decay in the waveform per cycle of oscillation. The measured damping of the waveform of Figure 11.7 is a factor of between four and five per cycle. This discrepancy between prediction and measurement may be due to the substantial mismatch between real transistors—a problem that has been completely neglected in the preceding analysis.

Recovery from Large Transients

When the feedback current is set below the stability limit, the circuit will recover from large transient inputs. The waveforms for V_2 and V_3 observed during such a recovery are shown in Figure 11.8. They are similar to those observed in the marginal-stability case (Figure 11.6). We can analyze the dynamics of the recovery using the construction shown in Figure 11.9. The slopes of all waveforms have been normalized to that for V_3. While V_2 is below the input voltage V_1, it rises with a slope of $2\alpha+1$. After V_2 exceeds V_1, its slope is $2\alpha-1$. The analysis is similar to that used to obtain the stability limit, except the excursion of V_h above V_1 is smaller than the initial deviation V_l below V_1. Because we have normalized all slopes to the slope of V_3, the total time will be V_l+V_h:

$$t_l + t_h = \frac{V_l}{2\alpha+1} + \frac{V_h}{2\alpha-1} = V_l + V_h \tag{11.16}$$

The decrement by which the amplitude decreases during each half-period is $\Delta V = V_l - V_h$. We can express that decrement in terms of the total peak-to-peak amplitude $V_l + V_h$. Solving Equation 11.16, we obtain

$$\frac{V_l - V_h}{V_l + V_h} = 4\alpha^2 - 2\alpha - 1 \tag{11.17}$$

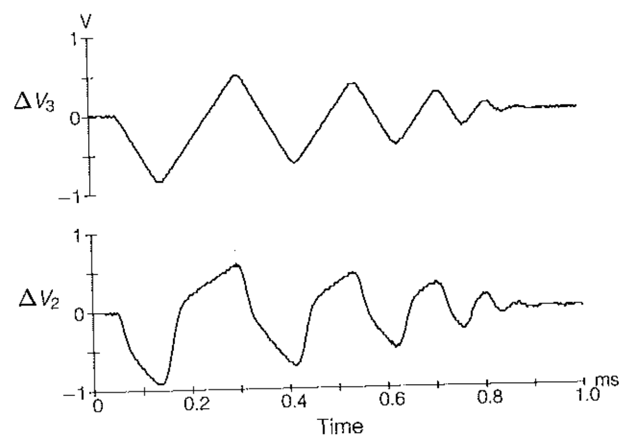

FIGURE 11.8 Recovery of the second-order section from a large-signal input. The response was measured for $\alpha < 0.8$.

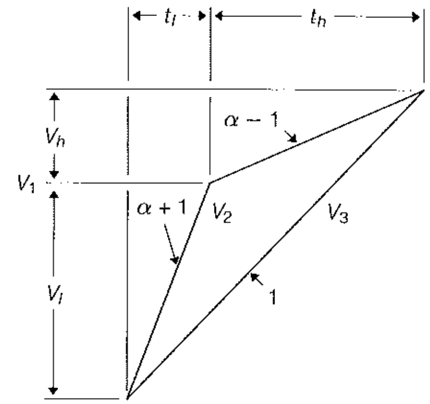

FIGURE 11.9 Idealized waveform of large-signal response for $\alpha < 0.809$. The amplitude of the oscillation decreases at each half-cycle because V_3 intersects V_2 slightly earlier on each excursion.

Setting this expression equal to 0 gives the value of α for marginal stability (Equation 11.14); larger values of α lead to growing solutions, smaller values lead to decaying solutions. Recognizing that the time for the half-period Δt is $V_l + V_h$, we see that the expression of Equation 11.17 is the *decrease in amplitude per unit time*:

$$\frac{\Delta V}{\Delta t} = 4\alpha^2 - 2\alpha - 1 \tag{11.18}$$

Equation 11.18 is the large-signal equivalent of a differential equation: It relates the amplitude at one time to that at a later time. Because the amplifiers are slew-rate limited, the value of the signal does not affect the rate, as it would in the "linear" regime. For this reason, the decay is linear instead of exponential; the large-signal behavior is simpler than the corresponding small-signal behavior. The approximately linear decay can be seen in Figure 11.8.

SUMMARY

We have derived the properties of second-order systems by way of a specific example. Second-order behavior can arise out of any complex system with feedback. We encountered a precursor to the matter in Chapter 10, when we observed damped oscillations in the response of the follower–differentiator circuit. It is common knowledge in the engineering community that feedback-control systems usually oscillate; getting these systems to be stable is the hardest part of designing them. This difficulty is the origin of the old electrical-engineering saying, "If you want an oscillator, design an amplifier."

Neural systems are notorious for generating large-signal limit-cycle behavior, either as part of their normal behavior pattern (as in a heartbeat) or as a pathology (as in an epileptic seizure). The deep mystery of large-scale neural systems is how they manage to stay stable at all! A hint can be gleaned from a comment by Gordon Shepherd "There is the impression from these experiments of a broad curtain of inhibition drawn across the olfactory bulb, through which excitation pierces, carrying specific information about the stimulating molecules." [Shepherd, 1979, p. 173] A similar comment could be made about any of the sensory systems, as well as about many other parts of the brain.

Inhibition as applied in biological systems is concerned with the magnitude of activity, not with the sign. A dark edge moving across the visual field excites as much response as a light edge would. The auditory system has a hard time distinguishing a negative pressure pulse from a positive one. It could be that the nonlinear nature of inhibitory feedback is the key to building a complex analog-processing system with a great deal of gain and time delay, and to keeping the entire mess stable. We will know that we understand these matters only when we can build such a system—and can keep it stable.

REFERENCES

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