

Problem 10.2

Given, the difference equation of the second-order system:

$$y[n] = 0.9(y[n-1] - y[n-2]) + x[n-1]$$

Taking z-transform: $Y(z)[1 - 0.9z^{-1} + 0.9z^{-2}] = z^{-1}X(z)$

$$\text{Hence, } H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 - 0.9z^{-1} + 0.9z^{-2}}$$

Let $z^{-1} = x$, then, $H(z = x^{-1}) = \frac{x}{1 - 0.9x + 0.9x^2}$. Considering, $F(z = x^{-1}) = \frac{1}{1 - 0.9x + 0.9x^2}$, the roots of the equation are: $x_1 = \frac{1}{2} + \frac{1.85}{2}j$ and

$x_2 = \frac{1}{2} - \frac{1.85}{2}j$. The product of the roots is $x_1x_2 = 0.9$.

$$\begin{aligned} \text{Then, } F(z = x^{-1}) &= \frac{1}{(x-x_1)(x-x_2)} = \frac{1}{x_1x_2(1-x/x_1)(1-x/x_2)} \\ &= \frac{0.9}{(1-x/x_1)(1-x/x_2)}, \quad 1/x_1 = 0.95e^{-j1.075} \text{ and } 1/x_2 = 0.95e^{j1.075} \end{aligned}$$

$$\text{Now, } F(z) = \frac{0.9}{(1-0.95e^{-j1.075}z^{-1})(1-0.95e^{j1.075}z^{-1})}$$

Expressing as partial fractions gives,

$$\begin{aligned} F(z) &= A/(1 - 0.95e^{-j1.075}z^{-1}) + B/(1 - 0.95e^{j1.075}z^{-1}) \\ \implies A(1 - 0.95e^{j1.075}z^{-1}) + B(1 - 0.95e^{-j1.075}z^{-1}) &= 0.9 \end{aligned}$$

Two equations can be formed:

$$\begin{aligned} A + B &= 0.9 \text{ and } -0.95z^{-1}[Ae^{j1.075} + Be^{-j1.075}] = 0, \implies \\ [Ae^{j1.075} + Be^{-j1.075}] &= 0 \end{aligned}$$

$A = 0.9 - B$. Substituting in the second equation yields,

$$B = \frac{0.9e^{j1.075}}{ej1.075 - e^{-j1.075}} = \frac{0.9e^{j1.075}}{2j \sin(1.075)} = \frac{0.9e^{j1.075}}{1.76e^{j\frac{\pi}{2}}} = 0.511e^{-j0.496}$$

$$A = 0.9 - B = \frac{-0.9e^{j1.075}}{ej1.075 - e^{-j1.075}} = 0.511e^{j0.496}.$$

Now, $F(z) = 0.511e^{j0.496}/(1 - 0.95e^{-j1.075}z^{-1}) + 0.511e^{-j0.496}/(1 - 0.95e^{j1.075}z^{-1})$

$$\begin{aligned} \text{Hence, } f[n] &= 0.511e^{j0.496}(0.95e^{-j1.075})^n u[n] + \\ &\quad 0.511e^{-j0.496}(0.95e^{+j1.075})^n u[n] \\ &= 0.511(0.95)^n [e^{j(1.075n - 0.496)} + e^{-j(1.075n - 0.496)}] \\ &= 1.02(0.95)^n \cos(1.075n - 0.496)u[n] \\ h[n] &= f[n-1] = 1.02(0.95)^{n-1} \cos(1.075(n-1) - 0.496)u[n-1] \end{aligned}$$

Thus, $h[n] = 0$ if $n < 1$ and $h[n] = 1.02(0.95)^{n-1} \cos(1.075(n-1) - 0.496)$ if $n \geq 1$.