Signals often are represented as a sum of sinusoids:

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \theta_k)$$

Sum of multiple sinusoids (same f)
→ single sinusoid (f)

For a periodic signal:
fundamental frequency = f_0 $\quad f_0 = \frac{1}{T_0}$

$$x(t) = A_0 + \frac{1}{2} \sum_{k=-1}^{\infty} A_k \cos(2\pi f_k t + \theta_k)$$

Fourier Series: Periodic signal in time
→ Discrete coefficient (f) samples

Expanding around orthogonal basis
(other basis possible)

Often we expand "cos" by complex exponentials

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi k f_0 t + \theta_k)}$$

+ & - Frequency

Operations on Fourier Series

$$\frac{dx(t)}{dt} = \sum_{k=-\infty}^{\infty} \underbrace{2\pi k f_0}_{= j\omega \;=\; j2\pi f} a_k e^{j(2\pi k f_0 t + \theta_k)}$$

(differentiation)

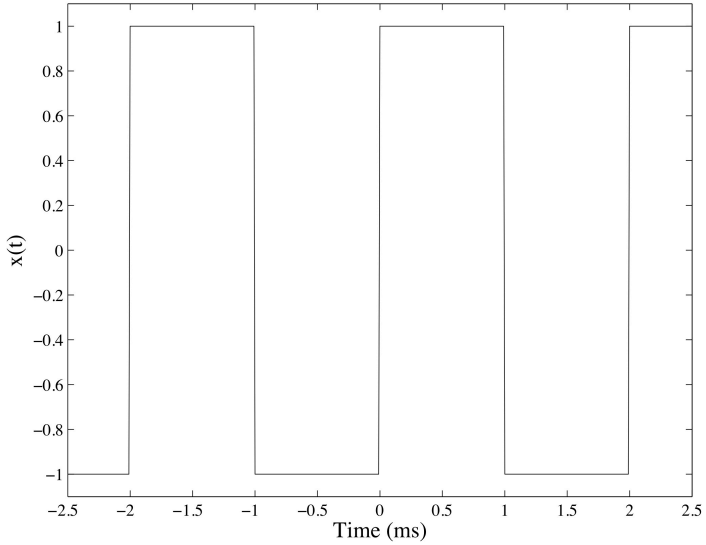
$$\int x(t)dt = \sum_{k=-\infty}^{\infty} \underbrace{\frac{a_k}{2\pi k f_0}}_{= 1/j\omega \;=\; 1/j2\pi f} e^{j(2\pi k f_0 t + \theta_k)}$$

(integration)

$$x(t - t_d) = \sum_{k=-\infty}^{\infty} a_k \underbrace{e^{-j2\pi k f_0 t_d}}_{= e^{-j2\pi f t_d}} e^{j(2\pi k f_0 t + \theta_k)}$$

(delay)

Fourier Series for a Square Wave



$$a_0 = f_0 \int_{-T_0/2}^{T_0/2} x(t)dt \to 0$$

$T_0 = 1/f_0 = 2ms$

$$a_k = 2f_0 \int_{-T_0/2}^{T_0/2} x(t) \sin(2\pi f_o k t)\, dt$$

(1ms)

(-1ms)

Integrate over a single period:

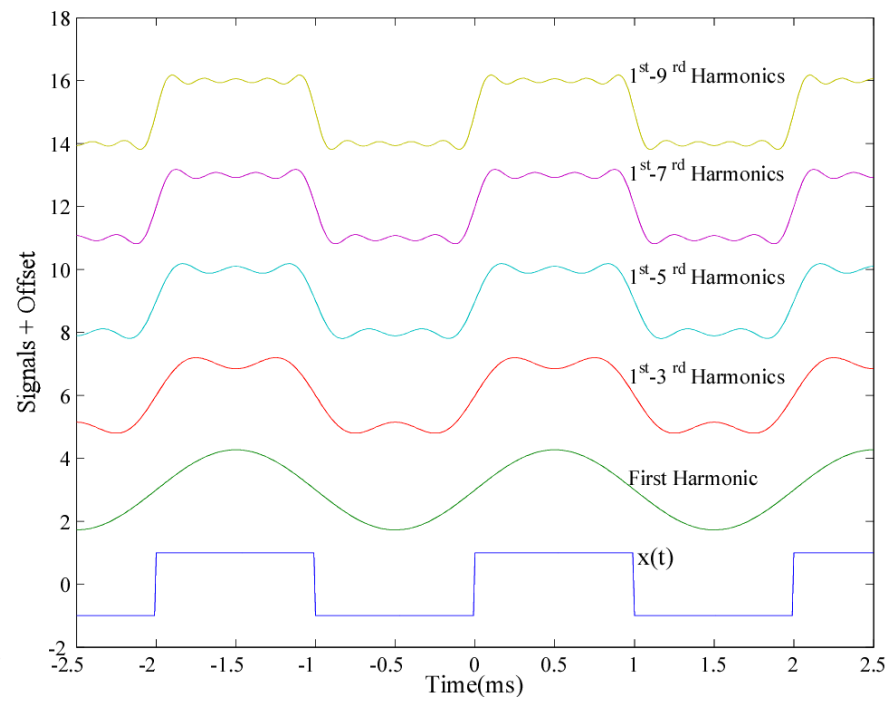
-1ms ($-T_0/2$) to 1ms ($T_0/2$)

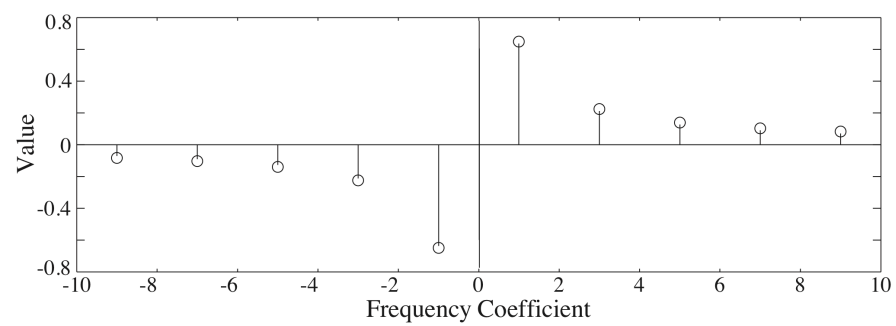
$$= 2f_0 \int_0^{T_0/2} \sin(2\pi f_o k t)\, dt - 2f_0 \int_{-T_0/2}^{0} \sin(2\pi f_o k t)\, dt$$

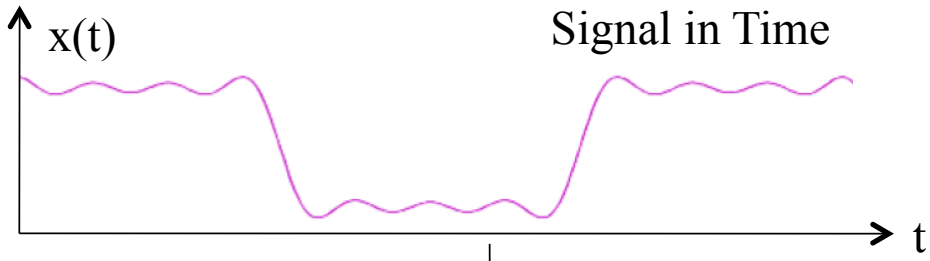
$$= 2\int_0^1 \sin(\pi k t)\, dt = \frac{2}{\pi k}(\cos(0) - \cos(k\pi))$$

(normalize time)

$$= \frac{4}{\pi k}$$ Odd k (= 0 for Even k)

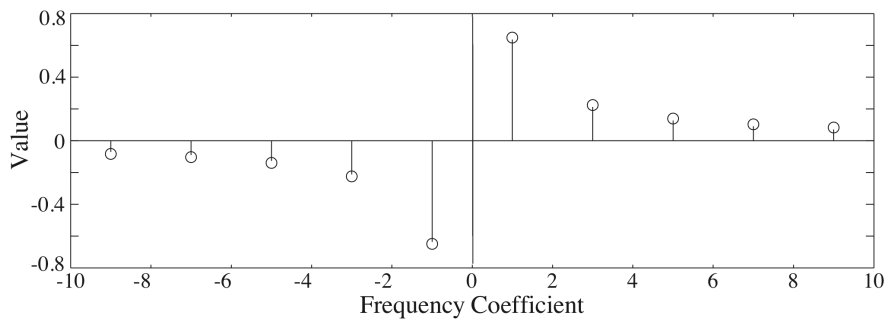




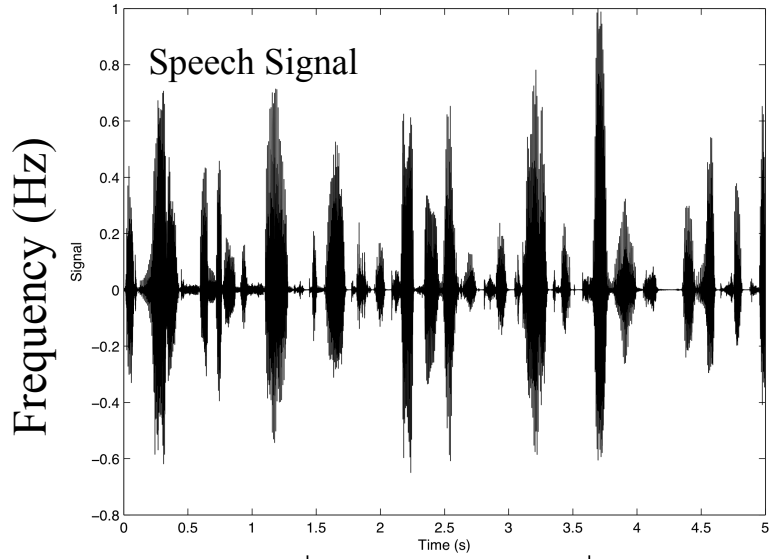


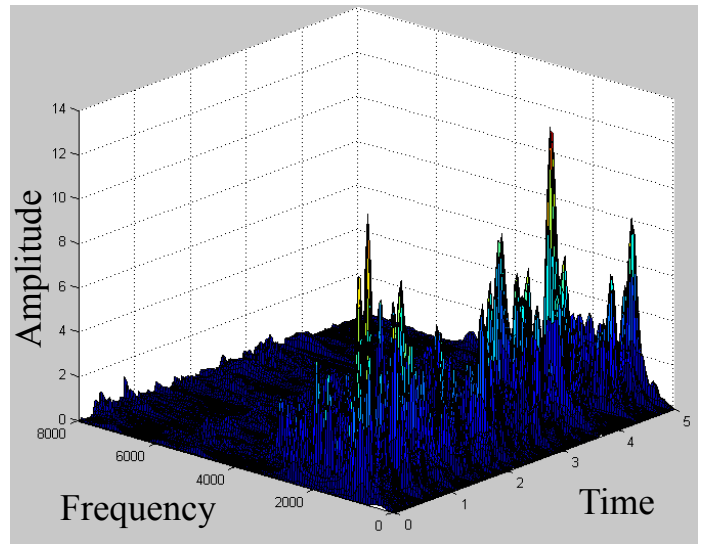
Fourier Series

Representation in Frequency



Generalizing the Frequency Representation (continuum)
→ Fourier Transform





Spectrogram: Time and Frequency
(& MATLAB command)