History of Starting with Signal Processing (SP) First

Classically (e.g. 1950s to 1990s),

- Electrical Engineering started with circuits
- Signal processing through circuits (historical start)

In the 1990s, Digital SP: 4th year or Grad course

Why not start with Signal Processing?

- Signal Processing first,
- then circuits enabled through SP

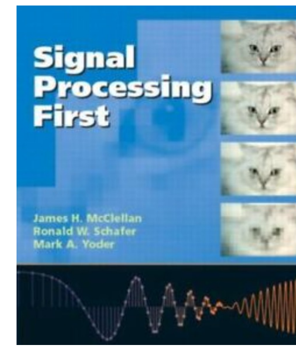
Ron Schafer

Jim McClellan

And others at GT, those who founded DSP
and some good friends at Rice, Rose-Hulman

Fall 1999 at GT:

- Full implementation
- Requirement for all ECE students



SP First (GT 1999)

Each week:

- Two large lecture sessions
- Faculty Led Recitation Session
- Weekly computer (MATLAB) exercises

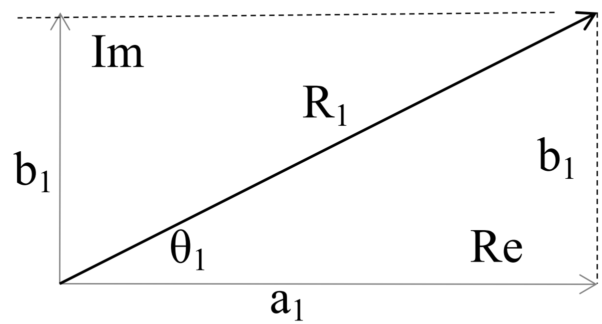
Complex Numbers = Real + j $\underbrace{\text{Imaginary}}_{\text{like a unicorn}}$

$$j = \sqrt{-1}$$

Unicorn → divide by "j" ⟶

$$C_1 = \underbrace{a_1 + jb_1}_{\text{(rectangular)}} = \underbrace{R_1 e^{j\theta_1}}_{\text{(polar)}}$$

$$C_2 = a_2 + jb_2 = R_2 e^{j\theta_2}$$



$$a_1 = R_1 \cos\theta_1$$

$$b_1 = R_1 \sin\theta_1$$

$$R_1 = \sqrt{a_1^2 + b_1^2}$$

$$\theta_1 = \tan^{-1}\left(\frac{b_1}{a_1}\right)$$

<u>Complex Operations</u>

Complex Addition: (rectangular)

$$C_1 + C_2 = (a_1 + jb_1) + (a_2 + jb_2) = a_1 + a_2 + j(b_1 + b_2)$$

Complex Multiplication: (polar)

$$C_1 C_2 = R_1 e^{j\theta_1} R_2 e^{j\theta_2} = R_1 R_2 e^{j(\theta_1 + \theta_2)}$$

One often converts between rectangular and polar forms

Euler's Formula:

$$\underline{e^{j\theta} = \cos\theta + j\sin\theta} \quad j = \sqrt{-1}$$

$$e^{j0} = e^{j2\pi} = -e^{j\pi} = 1$$

$$e^{j\pi/2} = -e^{j3\pi/2} = j$$

$$e^{j\theta} + e^{-j\theta} = 2\cos\theta \to \cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$e^{j\theta} - e^{-j\theta} = 2j\sin\theta \to \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$\begin{aligned}\cos^2(\theta) &= \left(\frac{e^{j\theta} + e^{-j\theta}}{2}\right)^2 \\ &= \frac{e^{j2\theta} + 2 + e^{-j2\theta}}{4} \\ &= \frac{1 + \cos(2\theta)}{2}\end{aligned}$$

$$\begin{aligned}\sin^2(\theta) &= \left(\frac{e^{j\theta} - e^{-j\theta}}{2j}\right)^2 \\ &= \frac{-e^{j2\theta} + 2 - e^{-j2\theta}}{4} \\ &= \frac{1 - \cos(2\theta)}{2}\end{aligned}$$

$$\begin{aligned}\cos(\theta)\sin(\theta) &= \left(\frac{e^{j\theta} + e^{-j\theta}}{2}\right)\left(\frac{e^{j\theta} - e^{-j\theta}}{2j}\right) \\ &= \frac{-e^{j2\theta} - e^{-j2\theta}}{4j} \\ &= -\sin(2\theta)/2\end{aligned}$$

$$\begin{aligned}\cos^3(\theta) &= \left(\frac{e^{j\theta} + e^{-j\theta}}{2}\right)^3 \\ &= \frac{1}{8}\left(e^{j3\theta} + 3e^{j\theta} + 3e^{-j\theta} + e^{-j3\theta}\right) \\ &= \frac{1}{8}\left(e^{j3\theta} + e^{-j3\theta} + 3e^{j\theta} + 3e^{-j\theta}\right) \\ &= \frac{3}{4}\cos(\theta) + \frac{1}{4}\cos(3\theta)\end{aligned}$$

$$\begin{aligned}\sin^3(\theta) &= \left(\frac{e^{j\theta} - e^{-j\theta}}{2j}\right)^3 \\ &= \frac{1}{8j}\left(e^{j3\theta} - e^{-j3\theta} - 3e^{j\theta} + 3e^{-j\theta}\right) \\ &= \frac{1}{8j}\left(e^{j3\theta} - 3e^{j\theta} + 3e^{-j\theta} - e^{-j3\theta}\right) \\ &= \frac{1}{4}\sin(3\theta) - \frac{3}{4}\sin(\theta)\end{aligned}$$

Sinusoidal functions

$$y_1(t) = \underbrace{A_1}_{\text{(Amplitude)}} \cos(2\pi f t + \underbrace{\theta_1}_{\text{(phase)}}) \rightarrow A_1 e^{j(2\pi f t + \theta_1)}$$

f = frequency of the sinusoid (Hz)

T = 1/f = period of the sinusoid (s)

2π f = radians of the sinusoid (rad)

$$Re\left\{A_1 e^{j(2\pi f t+\theta_1)}\right\} = Re\left\{A_1 \cos(2\pi f t + \theta_1) + j\sin(2\pi f t + \theta_1)\right\} =$$
$$= A_1 \cos(2\pi f t + \theta_1)$$

$$y_2(t) = A_2 \cos(2\pi f t + \theta_2) \rightarrow A_2 e^{j(2\pi f t + \theta_2)}$$

Summation ($y_1(t) + y_2(t)$):

(Same frequency)

$$\boxed{Re\left\{\sum_{k=1}^{n} X_k\right\} = \sum_{k=1}^{n} Re\left\{X_k\right\}}$$

$$\begin{aligned}
y_1(t) + y_2(t) &= A_1 \cos(2\pi f t + \theta_1) + A_2 \cos(2\pi f t + \theta_2) \\
&= Re\left\{A_1 e^{j(2\pi f t+\theta_1)} + A_2 e^{j(2\pi f t+\theta_2)}\right\} \\
&= Re\left\{\left(A_1 e^{j\theta_1} + A_2 e^{j\theta_2}\right) e^{j2\pi f t}\right\}
\end{aligned}$$

$$Ae^{j\theta} = A_1 e^{j\theta_1} + A_2 e^{j\theta_2}$$

$$y_1(t) + y_2(t) = Re\left\{Ae^{j\theta} e^{j2\pi f t}\right\} = A\cos(2\pi f t + \theta)$$

$$\begin{aligned}
y_1(t) + y_2(t) &= 5\cos(200t + \pi/3) \\
&\quad +5\sin(200t + \pi/6) \\
&= Re\left\{5e^{j(200t+\pi/3)} + 5e^{j(200t+2\pi/3)}\right\} \\
&= Re\left\{\left(5e^{j\pi/3} + 5e^{j2\pi/3}\right) e^{j200t}\right\} \\
&= Re\left\{\left(5\sqrt{3}e^{j\pi/2}\right) e^{j200t}\right\} \\
&= 5\sqrt{3}\cos(200t + \pi/2)
\end{aligned}$$

Linear Functions: f(x)

$$f(ax+by) = af(x) + bg(y)$$

$$f\left(ag(x) + bh(y)\right) = af\left(g(x)\right) + bf\left(g(y)\right)$$

<u>Example functions</u>

Differentiation

$$\frac{d}{dt}\left(ag(x) + bh(y)\right) =$$
$$a\frac{d}{dt}g(x) + b\frac{d}{dt}h(y)$$

Integration

$$\int \left(ag(x) + bh(y)\right) dt =$$
$$a\int g(x)dt + b\int h(y)dt$$

Linear gain factor:

$$f(ax) = af(x) \rightarrow \frac{d}{dt}\left(ag(x)\right) = a\frac{d}{dt}g(x), \int \left(ag(x)\right) dt = a\int g(x)dt$$

Linear (Time-Independent) Systems are characterized by sinusoids and exponentials

Single frequency input → Single frequency ouput

$$\sin(\omega t) \longrightarrow \boxed{\text{Linear System}} \longrightarrow H\sin(\omega t + \theta)$$

Signals often are represented as a sum of sinusoids:

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos\left(2\pi f_k t + \theta_k\right)$$

Sum of multiple sinusoids (same f)

→ single sinusoid (f)

For a periodic signal:

fundamental frequency = f_0 $\qquad f_0 = \dfrac{1}{T_0}$

$$x(t) = A_0 + \frac{1}{2}\sum_{k=-1}^{\infty} A_k \cos\left(2\pi f_k t + \theta_k\right)$$

Fourier Series: Periodic signal in time

→ Discrete coefficient (f) samples

Expanding around orthogonal basis

(other basis possible)

Often we expand "cos" by complex exponentials

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi k f_0 t + \theta_k)}$$

+ & - Frequency

Operations on Fourier Series

$$\frac{dx(t)}{dt} = \sum_{k=-\infty}^{\infty} \underbrace{2\pi k f_0}_{= j\omega = j2\pi f} a_k e^{j(2\pi k f_0 t + \theta_k)}$$

(differentiation)

$= j\omega$

$= j2\pi f$

$$\int x(t)dt = \sum_{k=-\infty}^{\infty} \underbrace{\frac{a_k}{2\pi k f_0}}_{= 1/j\omega} e^{j(2\pi k f_0 t + \theta_k)}$$

(integration)

$= 1/j\omega$

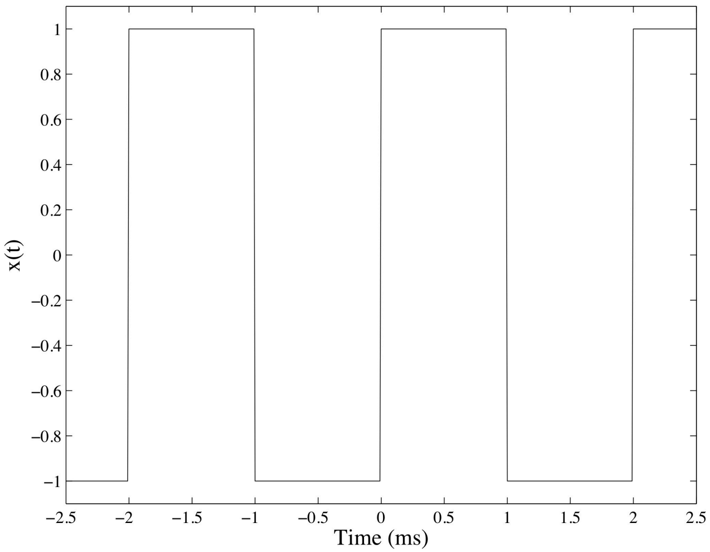
$= 1/j2\pi f$

$$x(t - t_d) = \sum_{k=-\infty}^{\infty} \underbrace{a_k e^{-j2\pi k f_0 t_d}}_{= e^{-j2\pi f t_d}} e^{j(2\pi k f_0 t + \theta_k)}$$

(delay)

$= e^{-j2\pi f t_d}$

Fourier Series for a Square Wave



$$T_0 = 1/f_0 = 2\text{ms}$$

$$a_0 = f_0 \int_{-T_0/2}^{T_0/2} x(t)dt \rightarrow 0$$

(1ms)

$$a_k = 2f_0 \int_{-T_0/2}^{T_0/2} x(t) \sin\left(2\pi f_o k t\right) dt$$

(-1ms)

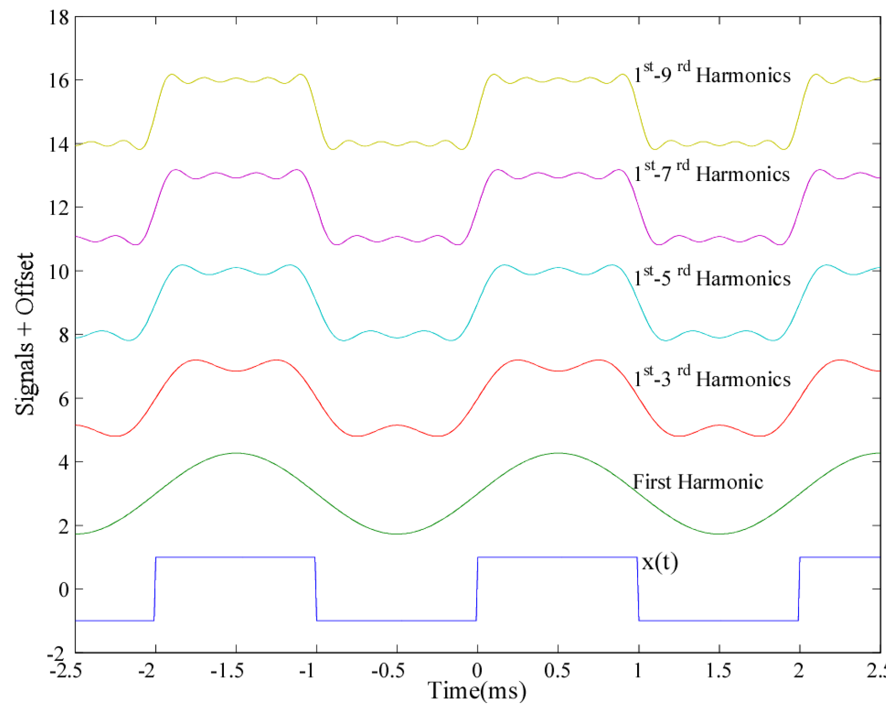
Integrate over a single period:
-1ms ($-T_0/2$) to 1ms ($T_0/2$)

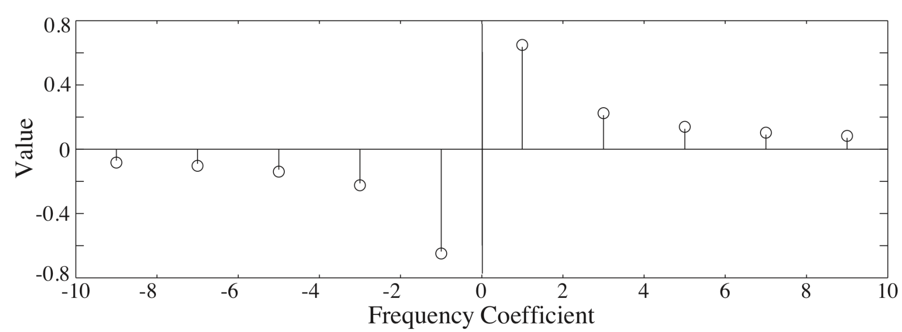
$$= 2f_0 \int_{0}^{T_0/2} \sin\left(2\pi f_o k t\right) dt - 2f_0 \int_{-T_0/2}^{0} \sin\left(2\pi f_o k t\right) dt$$

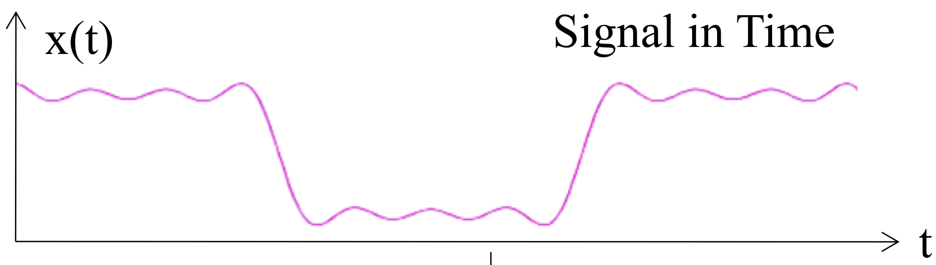
$$= 2 \int_{0}^{1} \sin\left(\pi k t\right) dt = \frac{2}{\pi k}\left(\cos(0) - \cos(k\pi)\right)$$

(normalize time)

$$= \frac{4}{\pi k}$$ Odd k (= 0 for Even k)

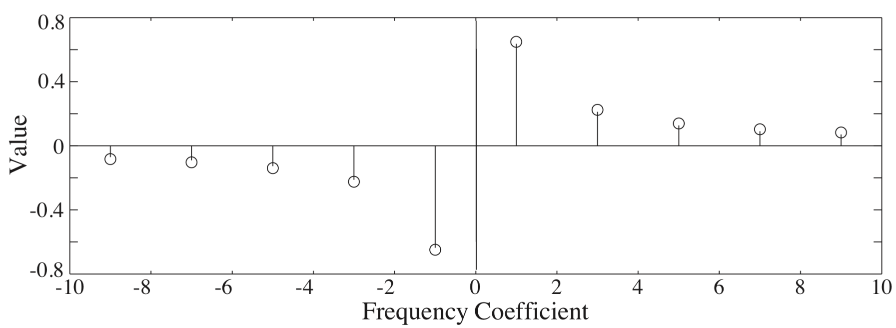




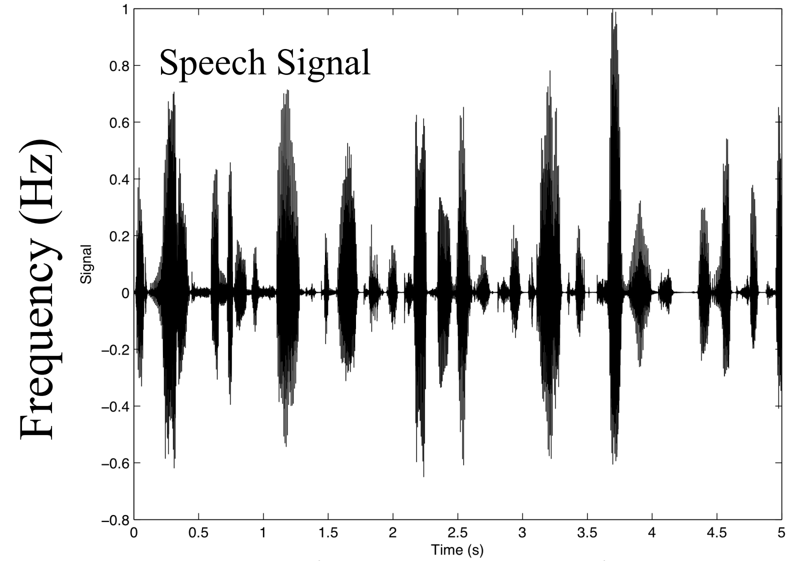


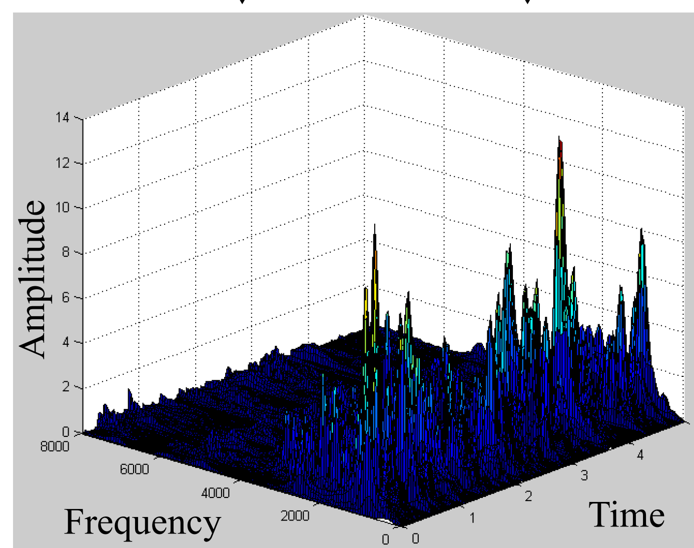
Fourier Series

Representation in Frequency



Generalizing the Frequency Representation (continuum)
→ Fourier Transform

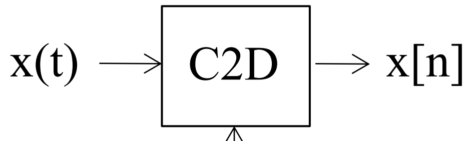


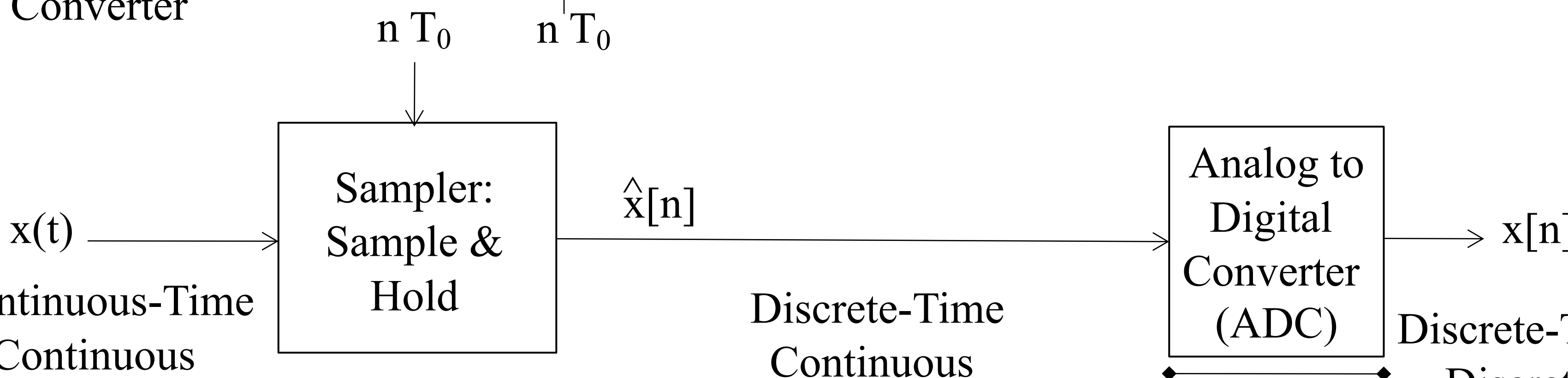


Spectrogram: Time and Frequency
(& MATLAB command)

Digital Sampling Implications

Continuous
To Digital
Converter

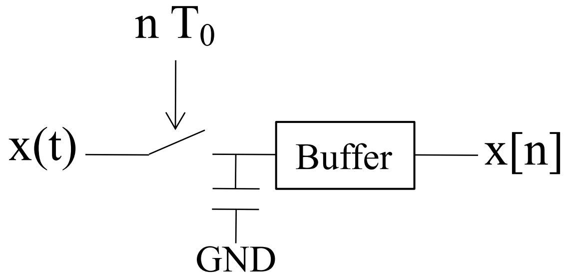


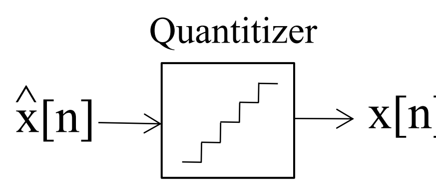


Continuous-Time
Continuous
Amplitude

Discrete-Time
Continuous
Amplitude

Discrete-Time
Discrete
Amplitude





Sample Time (T_0)
$f_0 = 1/T_0$ (sample frequency)

Make ideal clock (no jitter)
Make ideal sampler
Make ideal buffer

Limits SNR
& Linearity

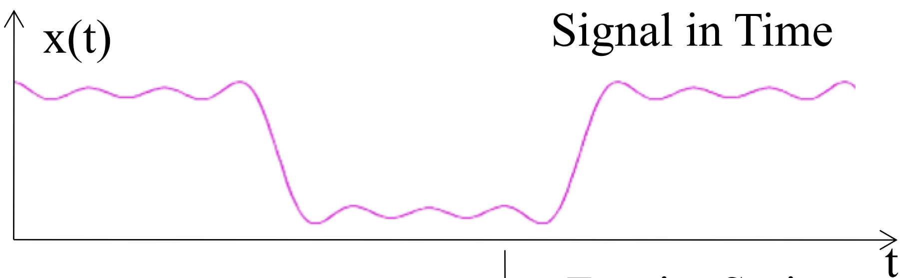
Comparitor(s)+ References
Typically “Assumed”
m-bit resolution

Limits SNR
& Linearity

Similar for D2C:
- Digital to Analog Converter (DAC)
- Analog Low-Pass Filter (Smoothing)

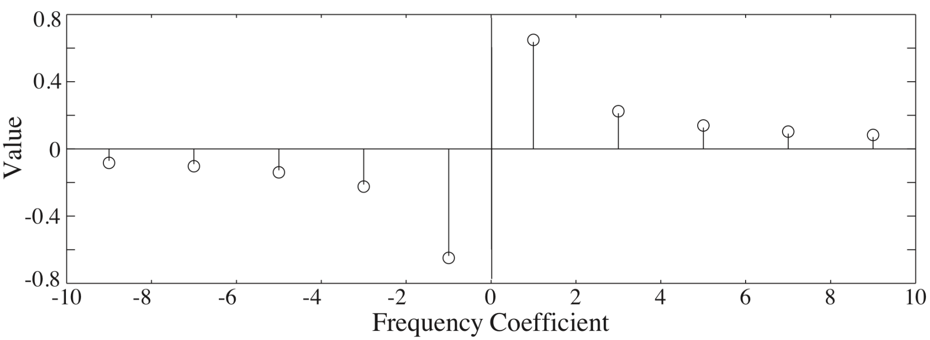
Periodic Signal in Time → Samples in Frequency

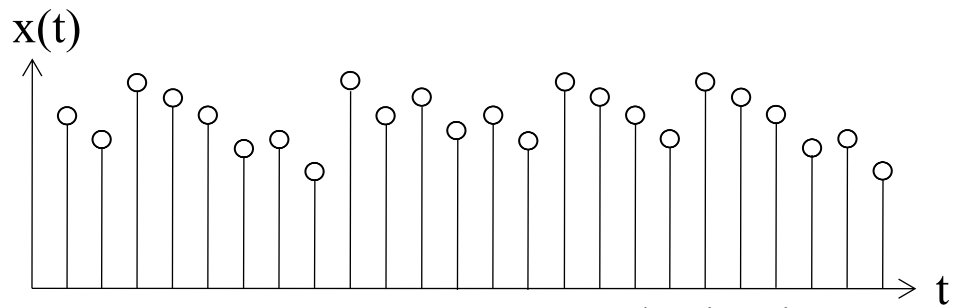
Samples in Time → Periodic Frequency Waveform



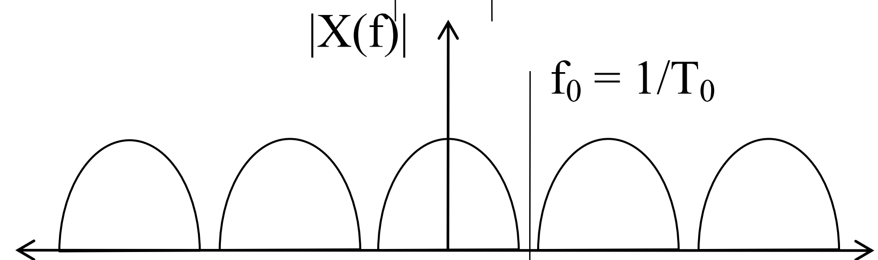
Fourier Series

Samples in Frequency





Samples in Time



Periodic Continuous Frequency

Nyquist: If sample 2x of highest frequency, then can one can perfectly reconstruct

Sampling of a single sinusoid:

$$x(t) = A\cos(\omega t + \theta), \quad \omega = 2\pi f$$

Sample: $f_s = \dfrac{1}{T_s}, \; t = nT_s$

$$x(t) = A\cos(\omega(nT_s) + \theta)$$

$$\hat{\omega} = \omega T_s$$

$$x(nT_s) \to x[n]$$

$$x[n] = A\cos(\hat{\omega} n + \theta)$$

$$\hat{\omega} = 2\pi f T_s$$

Can add or subtract
2π from $\hat{\omega}$
2π n comes out

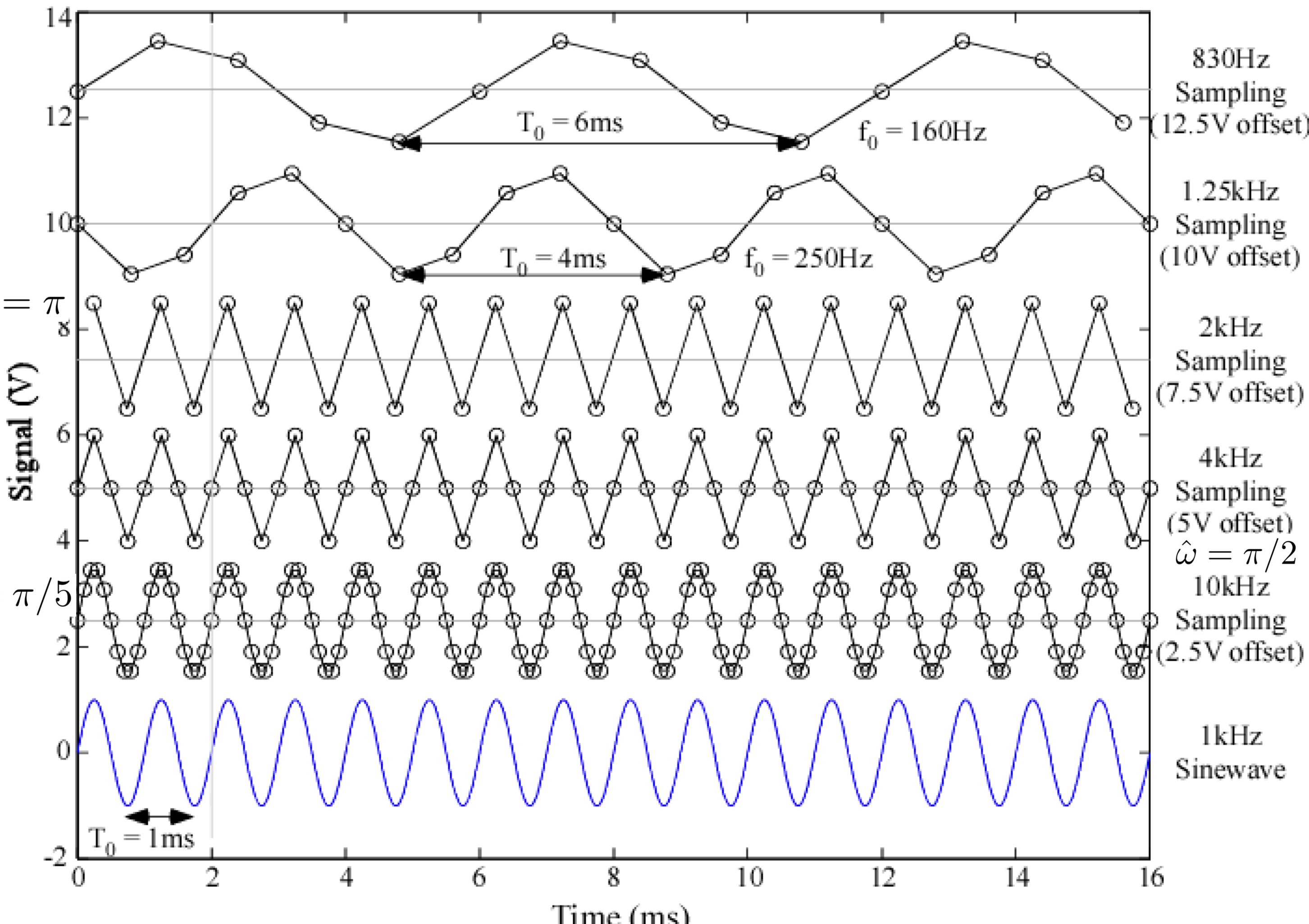
$\hat{\omega} = 2\pi(0.8) = -2\pi(0.2)$
At f_s=1.25kHz,
f → -0.2 x 1.25kHz
= -250Hz
(T_0 = 4ms)

$\hat{\omega} = 2\pi(1.2) = 2\pi(0.2)$
At f_s=830Hz,
f → 0.2 x 830Hz
= 166Hz
(T_0 = 6ms)

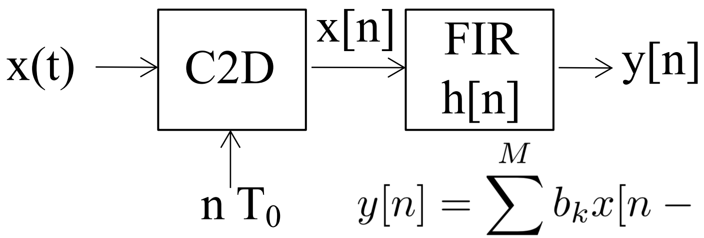
$\hat{\omega} = \pi$

$\hat{\omega} = \pi/5$

$\hat{\omega} = \pi/2$

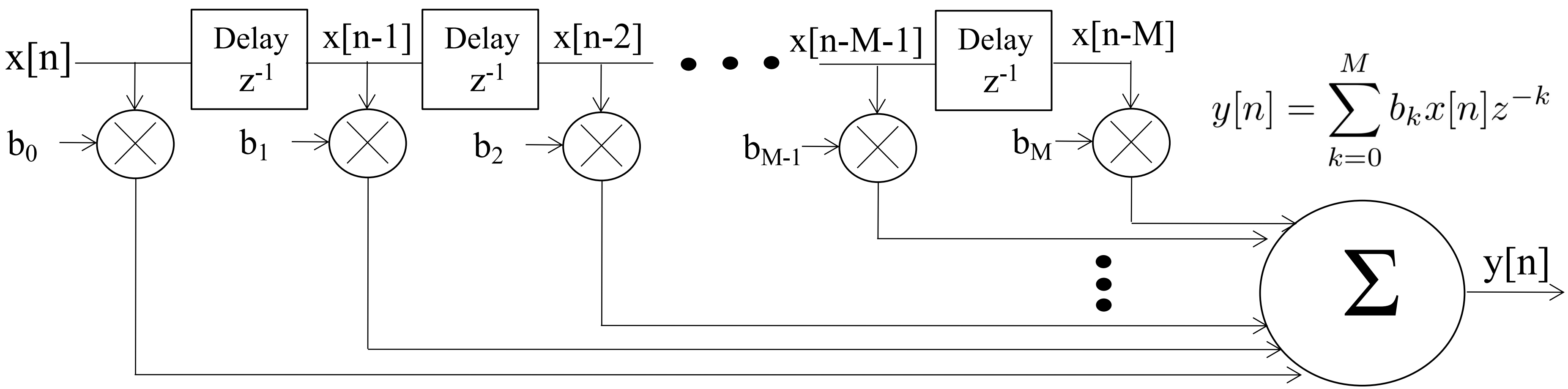


Finite Impulse Response Computation



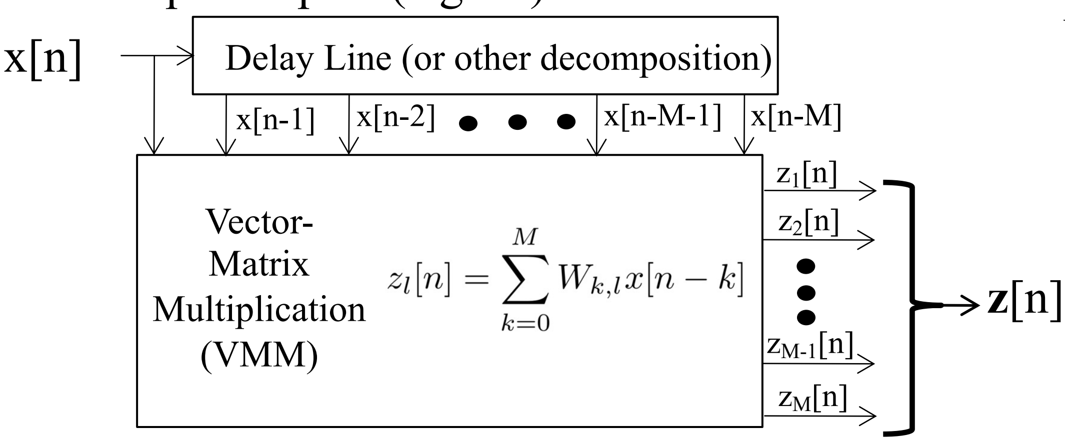
$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

$$y[n] = b[n] * x[n]$$



$$y[n] = \sum_{k=0}^{M} b_k x[n] z^{-k}$$

For multiple outputs (e.g. M):



$$z_l[n] = \sum_{k=0}^{M} W_{k,l} x[n-k]$$

VMM → Fundamental Computation

→ Multiply-Accumulates

Main Machine Learning Computation

Example other decompositions:

- Frequency Spectrum
- Wavelets / Subsample
- LC approximate delays

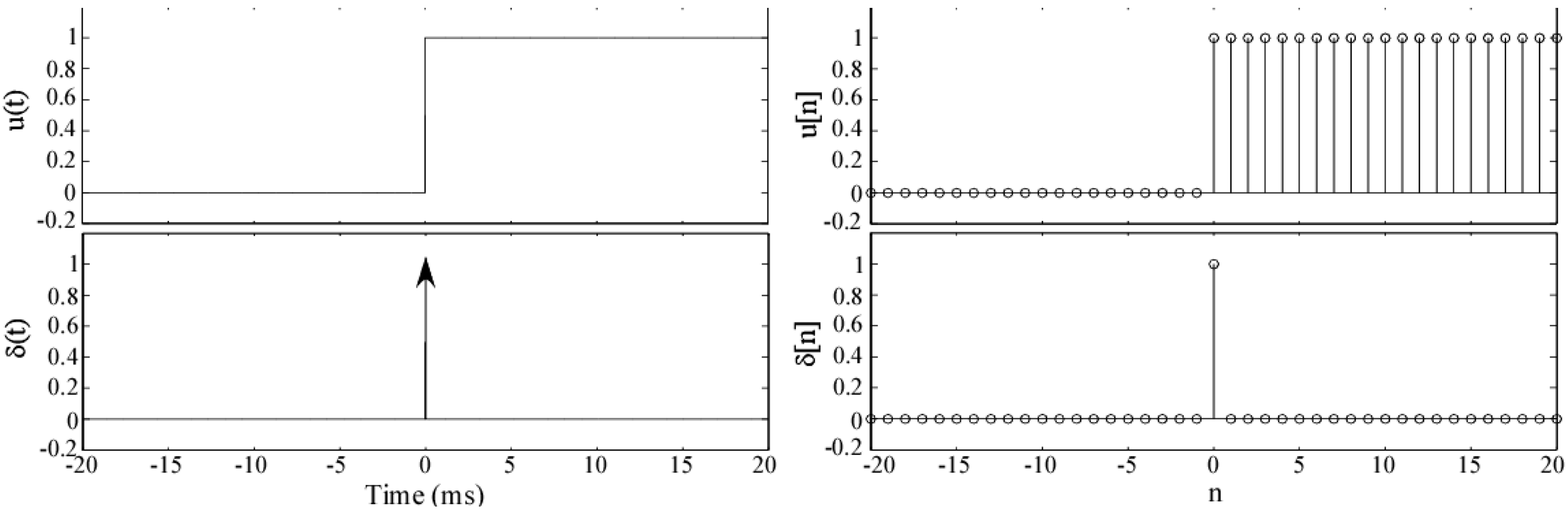
What if **x**(t) → **y**(t)?

Unit step function

• acts like flipping a switch at t=0

$$u(t) = \begin{cases} 1 & t >= 0 \\ 0 & t < 0 \end{cases} \quad \& \quad u[n] = \begin{cases} 1 & n >= 0 \\ 0 & n < 0 \end{cases}$$

Autonomous ODE: often parameter change & sometimes initial condition



Linear System Theory → Unit *Impulse* Function

• Derivative of Unit Step Function
• Signals composed of many Impulse Functions
• Often called an "Impulse"
• Autonomous ODE: mostly moving initial condition

Discrete time $\delta[n] = \begin{cases} 1 & n=0 \\ 0 & \text{otherwise} \end{cases}$

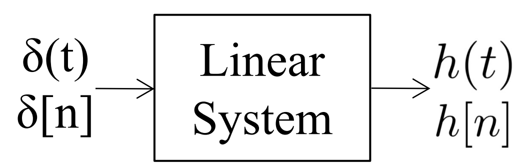
$\delta[n-3]$ = impulse delayed 3 time samples

What about $\delta(t)$, continuous-time impulse?

• mostly 0 everywhere
• approaching ∞ near t=0

Impulse Response:

Fundamental Linear Dynamics



Stable
Unstable
Oscillation

FIR
IIR

Linear Transforms:

Real Input Signals $\downarrow$ Transformed Input Signals $\longrightarrow$ Mathematical Operations (Transformed Signals) $\nearrow$ Real Output Signal

Direct Paths → Frequency Response

$\sin(\omega t)$, $\sin(\hat{\omega} n)$ $\longrightarrow$ Linear System $\longrightarrow$

$$|H|\sin(\omega t + \angle H(\omega))$$

$$|H|\sin(\hat{\omega}[n] + \angle H(\hat{\omega}))$$

Fourier → ω

Laplace: $s = \sigma + j\omega$

Z: $z^{-1} \rightarrow e^{-j\hat{\omega}n}$

Connections between transforms

s < 0 → | z | < 1 (unit circle)

Motivation: Solving Convolution (LTI: fixed coefficients)

Convolution in time (CT or DT) $\rightleftarrows$ Multiplication in transform space.

$$y[n] = h[n] * x[n] \qquad\qquad Y(z) = H(z)X(z)$$

$$\rightleftarrows \qquad Y(\omega) = H(\omega)X(\omega)$$

$$y[n] = h(t) * x(t) \qquad\qquad Y(s) = H(s)X(s)$$

Transforming Differential / Difference Equations to Algebraic Equations (then invert transform)

$$\frac{dy(t)}{dt} + y(t) \rightarrow sY(s) + Y(s)$$

$$x[n-2] + 2x[n-1] + x[n] \rightarrow X(z)z^{-2} + X(z)z^{-1} + X(z)$$

Frequency Response: Single Freq. Sinusoids

$$\sin(\omega t) \rightarrow \boxed{\text{Linear System}} \rightarrow |H|\sin(\omega t + \angle H(\omega))$$

Same frequency, different magnitude & phase

Discrete Time, FIR Filter:

$$x[n] = \sin(\hat{\omega} n) \qquad \rightarrow \boxed{\text{Linear System}} \rightarrow \qquad y[n] = |H|\sin(\hat{\omega}[n] + \angle H(\hat{\omega}))$$

$$x[n] = e^{j\hat{\omega} n} \qquad y[n] = b[n] * x[n] \qquad y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

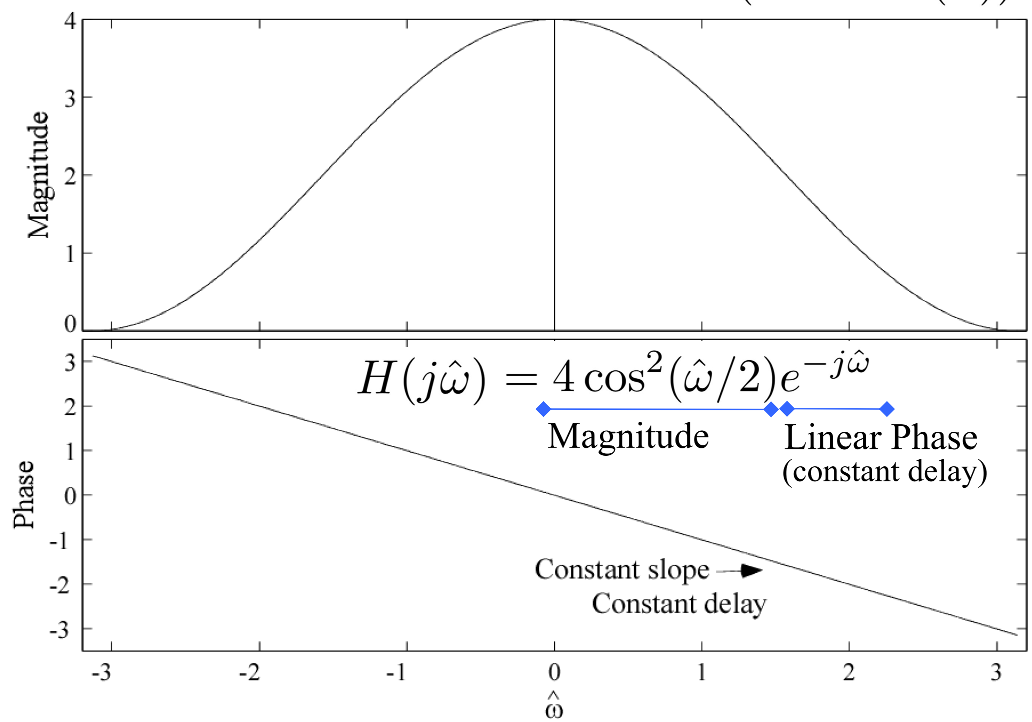
$$\mathbf{b} = [1 \;\; 2 \;\; 1]$$

$$\begin{aligned} H(j\hat{\omega}) &= 1 + 2e^{-j\hat{\omega}} + e^{-j2\hat{\omega}} \\ &= e^{-j\hat{\omega}}\left(e^{j\hat{\omega}} + 2 + e^{-j\hat{\omega}}\right) \\ &= e^{-j\hat{\omega}}\left(2 + 2\cos(\hat{\omega})\right) \end{aligned}$$

$$\begin{aligned} y[n] &= \sum_{k=0}^{M} b_k e^{j\hat{\omega}(n-k)} \\ &= e^{j\hat{\omega} n} \sum_{k=0}^{M} b_k e^{-j\hat{\omega} k} \\ &= x[n] \sum_{k=0}^{M} b_k e^{-j\hat{\omega} k} \\ &= x[n] H(j\hat{\omega}) \end{aligned}$$

$$-\pi < \hat{\omega} < \pi$$

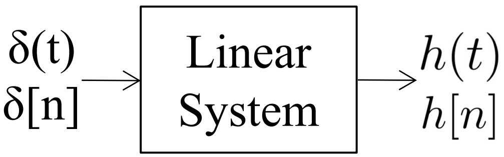
(Nyquist)

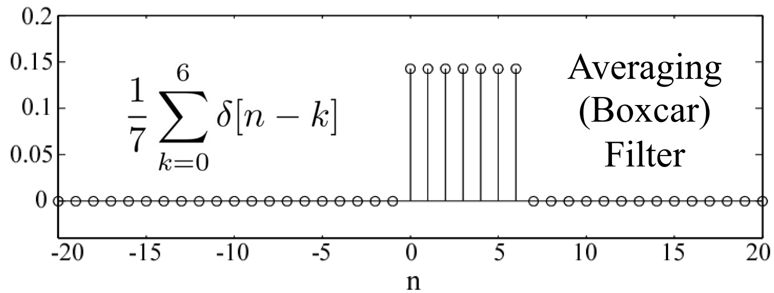


Convolution (Discrete or Continuous):

Solving for a linear system response
to an arbitrary waveform
by decomposing the input signal
into several impulse functions

Impulse Response:





Any input signal
Is a set of impulses
(DT or CT)

Convolution: Enables solution of any Linear Time-Invariant System With an arbitrary input

$$y[n] = h[n] * x[n]$$

$$y[n] = \sum_{k=0}^{M} h_k x[n-k]$$

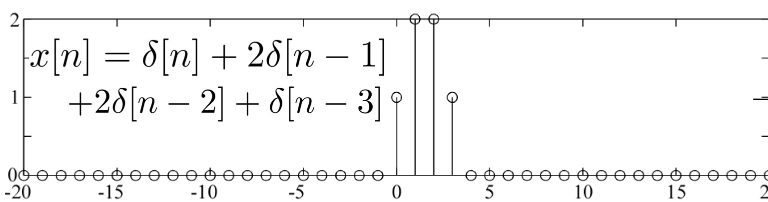
$$y[t] = h(t) * x(t)$$

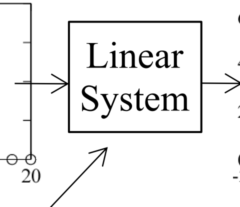
$$y(t) = \int_{-\infty}^{\infty} h(t-t_1)x(t_1)dt_1$$

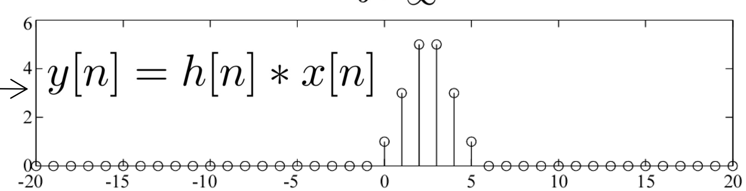
$$= \int_{-\infty}^{\infty} x(t-t_1)h(t_1)dt_1$$

Discrete-Time is easier to understand / visualize

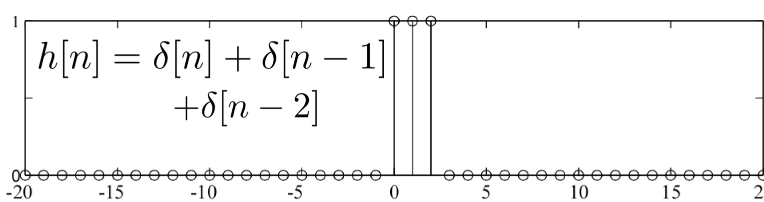
Input x[n]







Impulse Response h[n]



Two Impulse Convolution: $\delta[n-2] * \delta[n-3] = \delta[n-5]$

$$y[n] = (\delta[n]+\delta[n-1]+\delta[n-2])(\delta[n]+2\delta[n-1]+2\delta[n-2]+\delta[n-3])$$
$$= \delta[n]+3\delta[n-1]+5\delta[n-2]+5\delta[n-3]+3\delta[n-4]+\delta[n-5]$$

Relationship to multiplying polynomials

Boxcar Filter → Averaging Filter

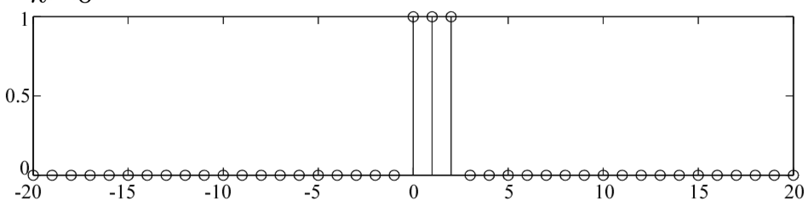
$$y[n] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-k]$$

A typical Low-Pass Filter to remove "noise" and unwanted variations

M=3 Sample Averaging

$$y[n] = \frac{1}{3}\sum_{k=0}^{2} x[n-k]$$

Impulse Response



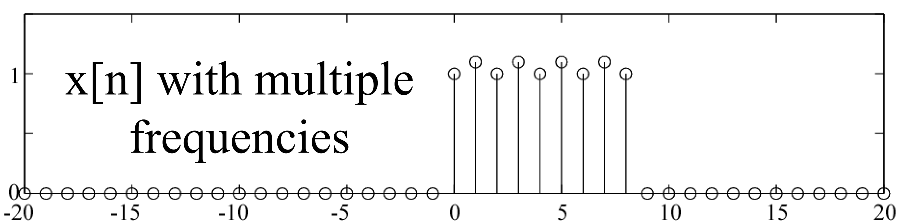
Frequency Response

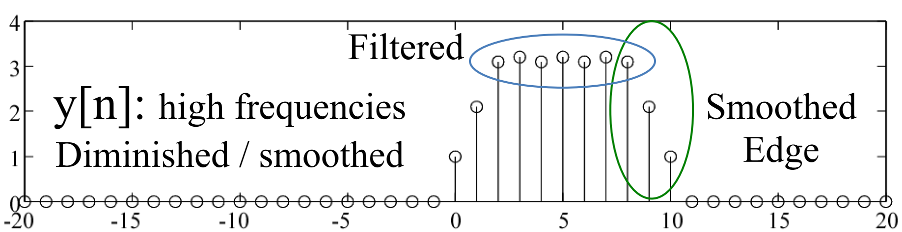
$$H(j\hat{\omega}) = \frac{1}{3}\left(1 + e^{-j\hat{\omega}} + e^{-j2\hat{\omega}}\right)$$

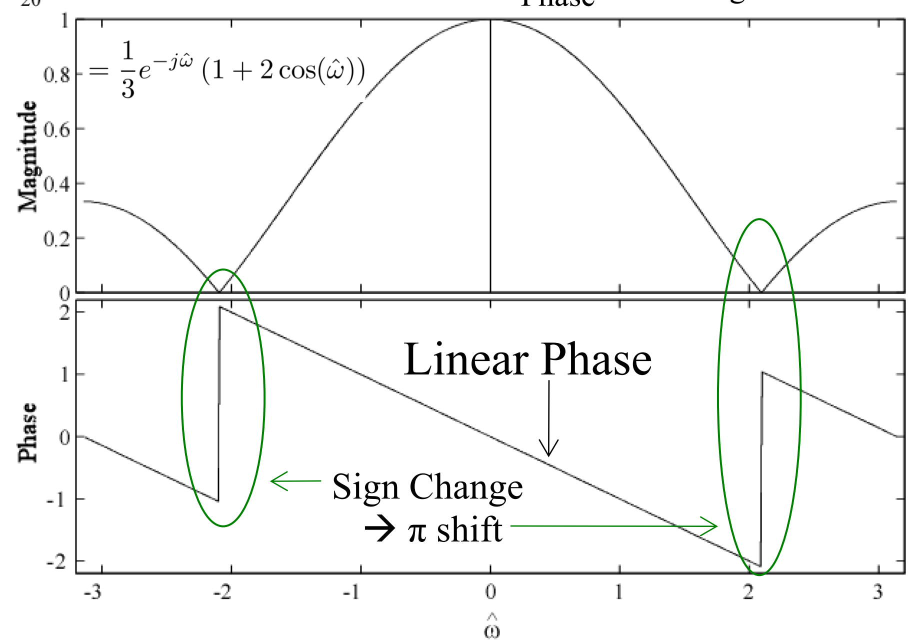
$$= \frac{1}{3}\left(1 + e^{-j\hat{\omega}} + e^{-j2\hat{\omega}}\right)$$

$$= \underbrace{\frac{1}{3}e^{-j\hat{\omega}}}_{\text{Phase}}\underbrace{\left(e^{j\hat{\omega}} + 1 + e^{-j\hat{\omega}}\right)}_{\text{Magnitude}}$$

$$y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2])$$







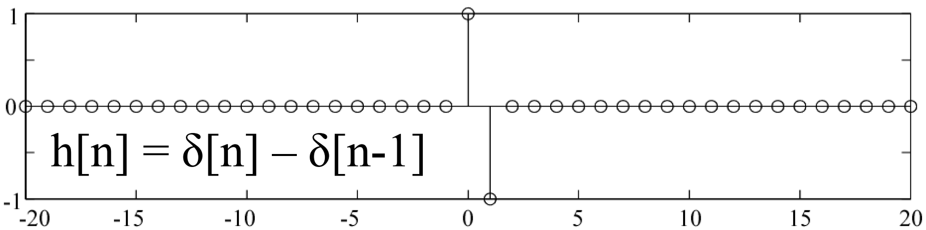
Differencing FIR Filter →

High-Pass based on a Derivative

$$y(t) = \frac{dx(t)}{dt} \approx \frac{x(t) - x(t-\Delta)}{\Delta}$$

$$\rightarrow (x[n] - x[n-1])$$

$$y[n] = x[n] - x[n-1]$$

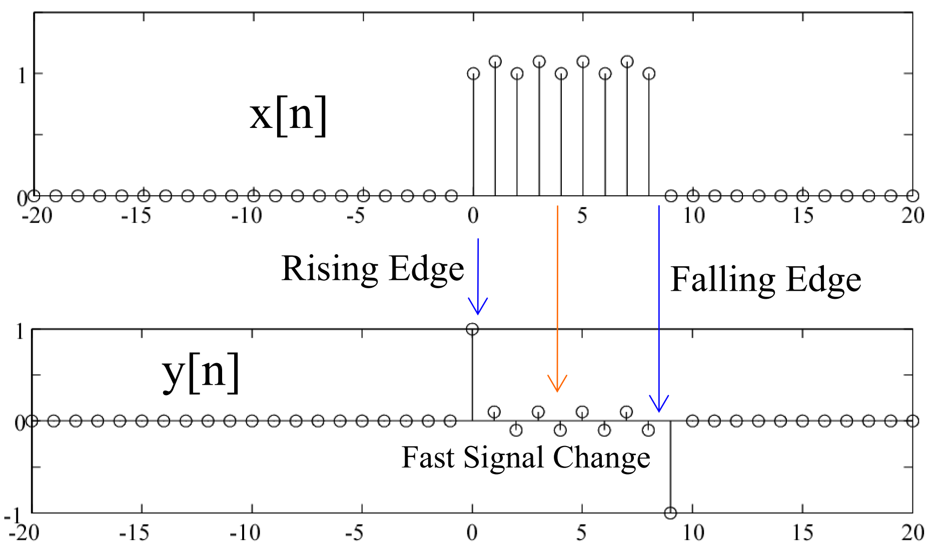


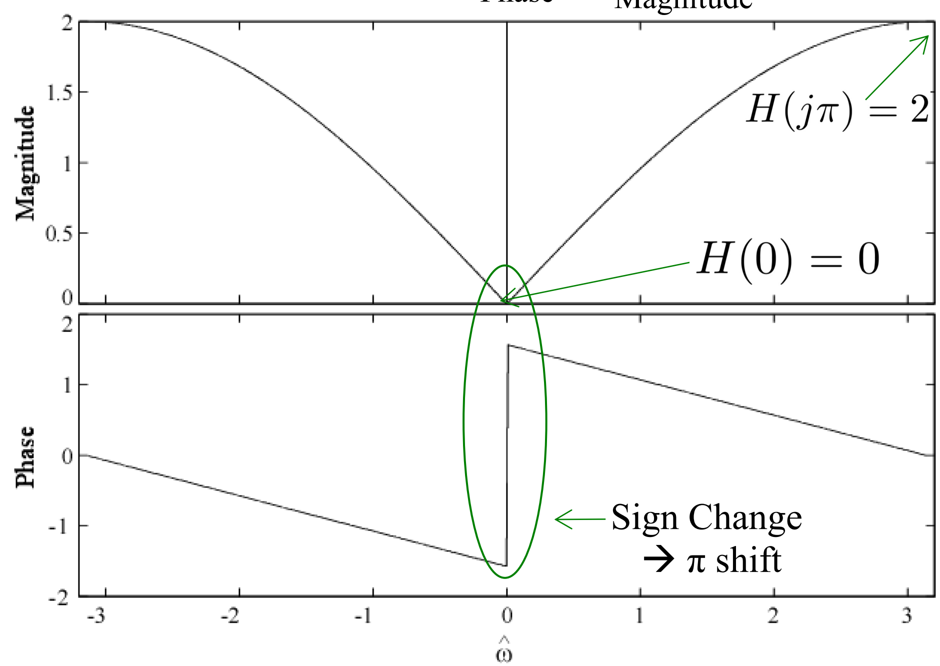
$$H(j\hat{\omega}) = 1 - e^{-j\hat{\omega}}$$

$$= e^{-j\hat{\omega}/2}\left(e^{j\hat{\omega}/2} - e^{-j\hat{\omega}/2}\right)$$

$$= \underbrace{2je^{-j\hat{\omega}/2}}_{\text{Phase}}\ \underbrace{\sin(\hat{\omega}/2)}_{\text{Magnitude}}$$

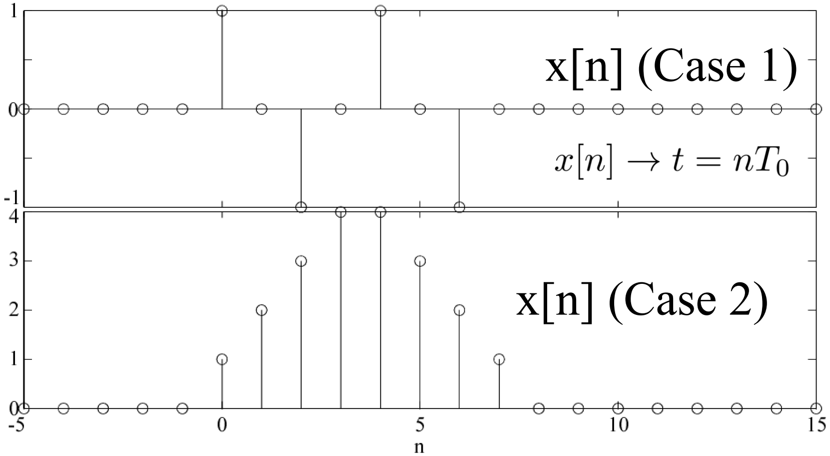
Approximate Derivative on input signal (x[n]):





$$y[n] = -x[n] + 2x[n-1] - x[n-2]$$

$$h[n] = -\delta[n] + 2\delta[n-1] - \delta[n-2]$$



$$y[n] = h[n] * x[n] \qquad \text{Convolution}$$

Case 1: $x[n] = \delta[n] - \delta[n-2] + \delta[n-4] - \delta[n-6]$

$$y[n] = (-\delta[n] + 2\delta[n-1] - \delta[n-2])$$
$$(\delta[n] - \delta[n-2] + \delta[n-4] - \delta[n-6])$$
$$= -\delta[n] + 2\delta[n-1] - 2\delta[n-3]$$
$$+2\delta[n-5] - 2\delta[n-7] + \delta[n-8]$$

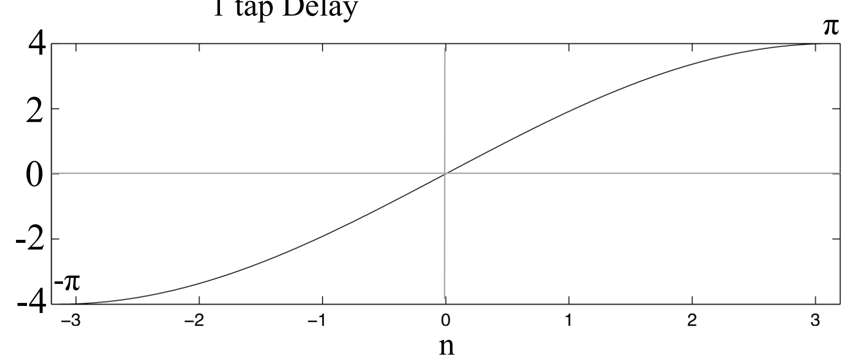
Case 2: make a table

n=0	1	2	3	4	5	6	7	8	9	10
-1	-2	-3	-4	-4	-3	-2	-1			
	2	4	6	8	8	6	4	2		
		-1	-2	-3	-4	-4	-3	-2	-1	
-1	0	0	0	1	1	0	0	0	-1	

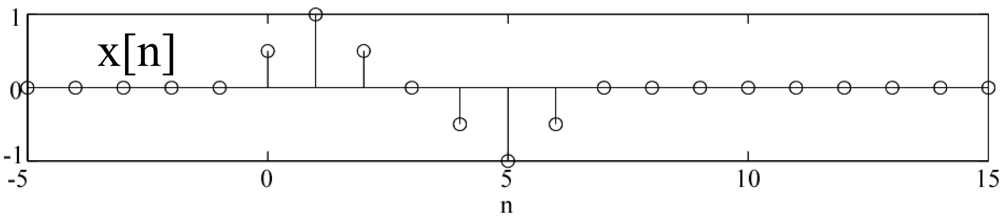
$$y[n] = -\delta[n] + \delta[n-4] + \delta[n-5] - \delta[n-9]$$

Frequency Response of h[n]

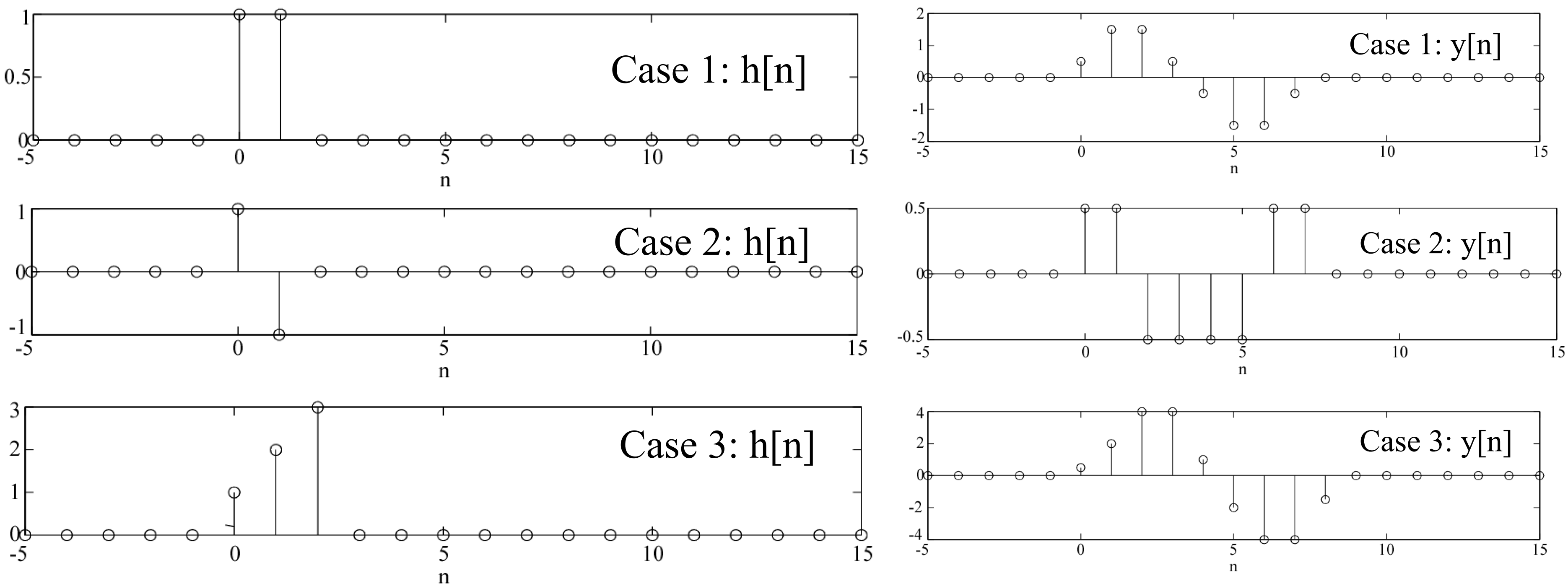
$$H(\hat{\omega}) = \sum_{k=0}^{2} h_k e^{-j\hat{\omega}k}$$
$$= -1 + 2e^{-j\hat{\omega}} - e^{-j2\hat{\omega}}$$
$$= e^{-j\hat{\omega}}\left(-e^{j\hat{\omega}} + 2 - e^{-j\hat{\omega}}\right)$$
$$= e^{-j\hat{\omega}}\left(2 - 2\cos(\hat{\omega})\right)$$
$$= \underbrace{e^{-j\hat{\omega}}}_{\substack{\text{Phase} \\ \text{1 tap Delay}}}\underbrace{\left(4\sin(\hat{\omega}/2)\right)}_{\text{Magnitude}}$$



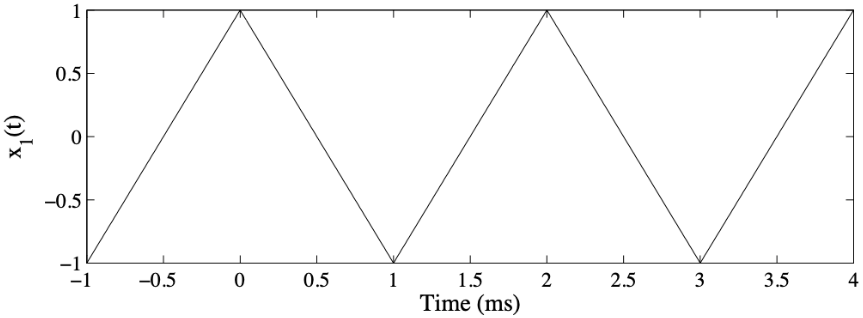
Convolution Examples

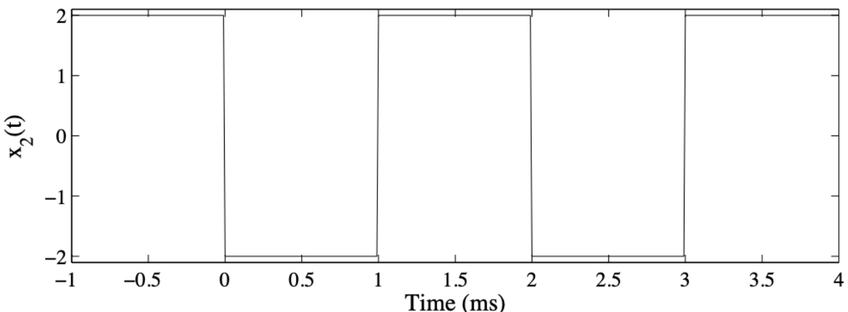


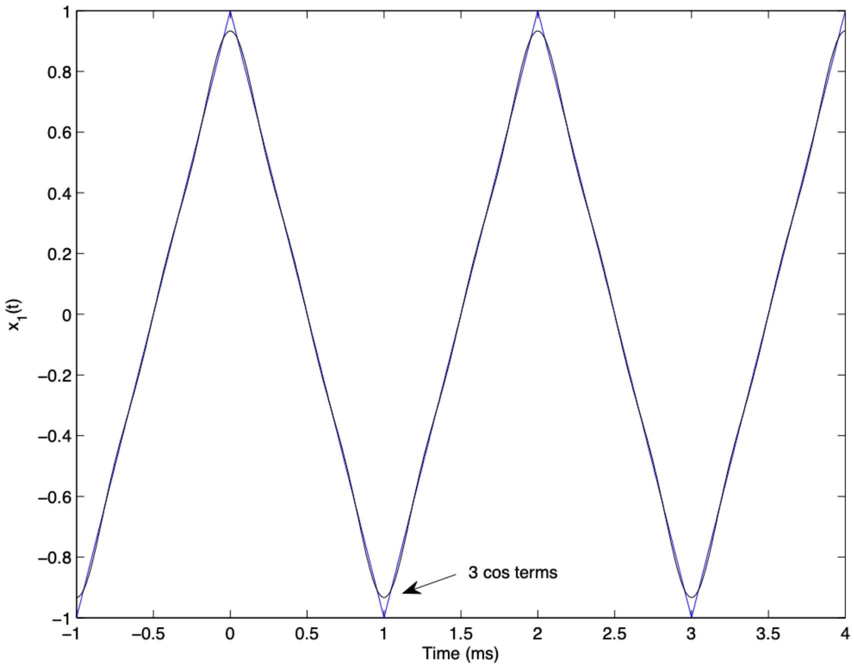
h[n] $\longrightarrow$ $y[n] = h[n] * x[n]$ $\longrightarrow$ y[n]



Fourier Series for Triangle Waveform







$T_0 = 1/f_0 = 2\text{ms}$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi k f_0 t + \theta_k)}$$

$$\int x(t)dt = \sum_{k=-\infty}^{\infty} \underbrace{\frac{a_k}{2\pi k f_0}}_{= 1/j\omega = 1/j2\pi f} e^{j(2\pi k f_0 t + \theta_k)}$$

(integration)

$$x_1(t) = \frac{8}{\pi^2}\cos(2\pi f_0 t) + \frac{8}{9\pi^2}\cos(6\pi f_0 t) + \frac{8}{25\pi^2}\cos(10\pi f_0 t) + O(7)$$

$$x_2(t) = -\sum_{k=1}^{\infty} \frac{8}{\pi k}\sin(2\pi f_0 k t)$$

k odd

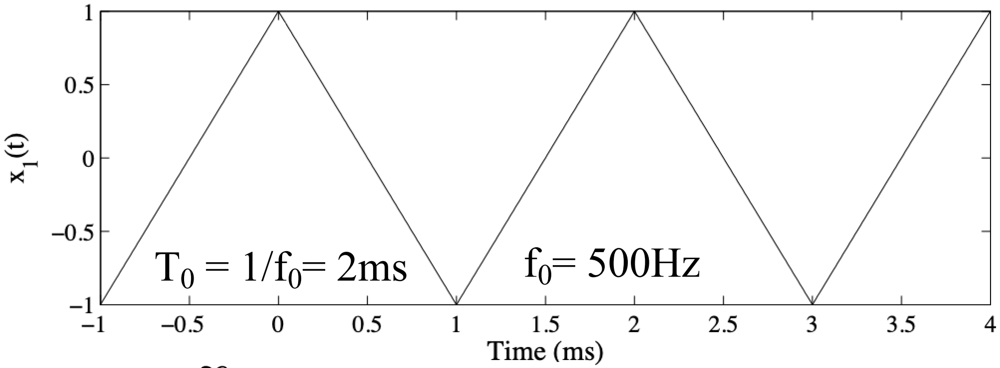
Amplitude is 2 → -2

$$x_2(t) = T_0 \frac{dx_1}{dt}$$

$$x_1(t) = \sum_{k=1}^{\infty} \frac{8}{(\pi k)^2}\cos(2\pi f_0 k t)$$

k odd,
= 0 k even

Sampling & Operating on a Triangle Waveform

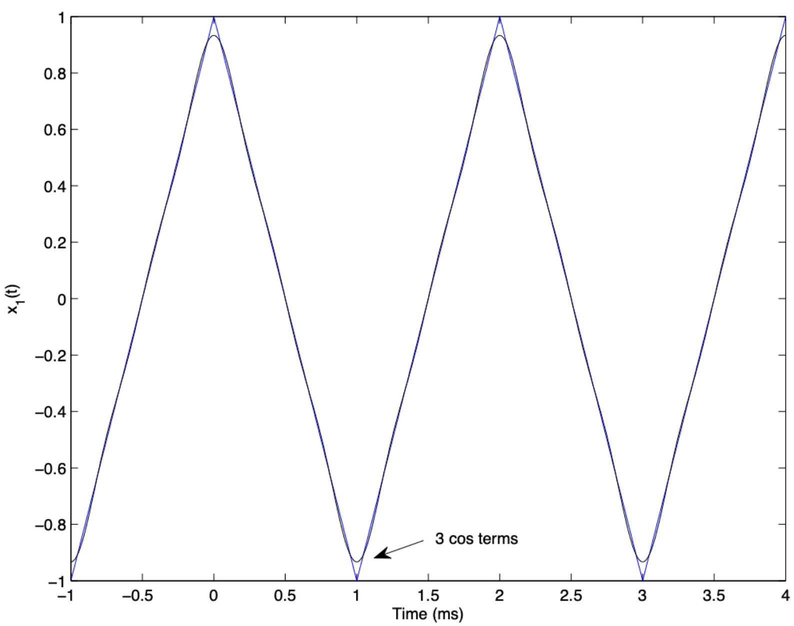


$$x_1(t) = \sum_{k=1}^{\infty} \frac{8}{(\pi k)^2} \cos(2\pi f_0 k t)$$ Odd k (= 0 for Even k)

1st order: $\frac{8}{\pi^2}$ 7th order: $\frac{8}{49\pi^2}$ ~ 2% Signal

First three terms have 98% of signal

$$x_1(t) = \frac{8}{\pi^2}\cos(2\pi f_0 t) + \frac{8}{9\pi^2}\cos(6\pi f_0 t) + \frac{8}{25\pi^2}\cos(10\pi f_0 t) + O(7)$$



For 2-4% error, sample at 5kHz (Nyquist) to get the 5th order term

$$\hat{\omega} = \frac{\pi}{5}, \frac{3\pi}{5}, \pi$$

The sampled signal then goes through

$$y[n] = x[n] - x[n-1]$$

Frequency Response

$$\begin{aligned} H(j\hat{\omega}) &= 1 - e^{-j\hat{\omega}} \\ &= e^{-j\hat{\omega}/2}\left(e^{j\hat{\omega}/2} - e^{-j\hat{\omega}/2}\right) \\ &= \underbrace{2je^{-j\hat{\omega}/2}}_{\text{Phase}}\underbrace{\sin(\hat{\omega}/2)}_{\text{Magnitude}} \end{aligned}$$

$\hat{\omega}$	Gain	Value
$\pi/5$	0.31	0.25
$3\pi/5$	0.81	0.073
π	1	0.032

Addition of Sinusoids of same frequency

$$y_1(t) = \underbrace{A_1}_{\text{(Amplitude)}} \cos(2\pi f t + \underbrace{\theta_1}_{\text{(phase)}}) \rightarrow A_1 e^{j(2\pi f t + \theta_1)}$$

$$Re\left\{\sum_{k=1}^{n} X_k\right\} = \sum_{k=1}^{n} Re\left\{X_k\right\}$$

$$\underbrace{4\cos(\omega_1 t)}_{4e^{j0}} + \underbrace{3\sin(\omega_1 t)}_{3e^{-j\pi/2}} \quad = 3\cos(\omega_1 t - \pi/2)$$

Magnitude = 5, phase = -0.644 → $5\cos(\omega_1 t - 0.644)$

$$5\cos(\omega_1 t + \pi/3) + 5\cos(\omega_1 t - \pi/3)$$

$$5e^{j\pi/3} + 5e^{-j\pi/3} = 10\cos(\pi/3) = 5$$

$5\cos(\omega_1 t)$

$$2\cos(\omega_1 t + \pi/6) - 2\cos(\omega_1 t - \pi/6)$$

$$2e^{j\pi/6} - 2e^{-j\pi/6} = 4j\sin(\pi/6) = 2j$$

$2\cos(\omega_1 t + \pi/2)$

Review of Initial Digital Signal Processing Concepts

T_0 Periodic Waveform Period

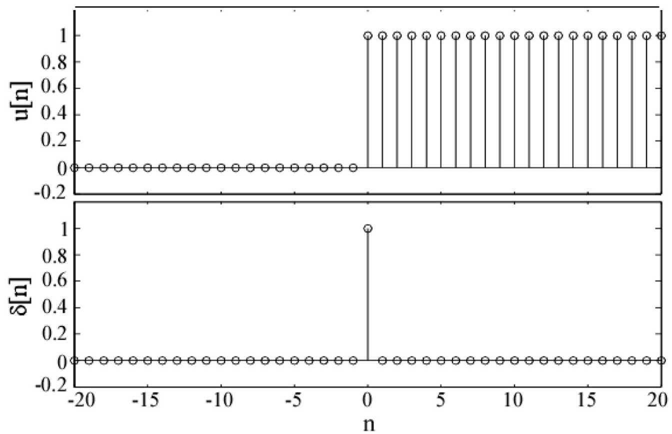
$f(nT_s)$ Sampled function $\quad f_s = \dfrac{1}{T_s}$ Sampling Period

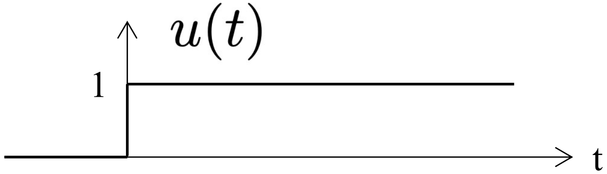
Discrete Step Function

$$u[n] = \begin{cases} 1 & n >= 0 \\ 0 & n < 0 \end{cases}$$

Discrete Impulse Function

$$\delta[n] = \begin{cases} 1 & n=0 \\ 0 & \text{otherwise} \end{cases}$$





$$u(t) = \begin{cases} 1 & t >= 0 \\ 0 & t < 0 \end{cases}$$

Unit Step Response (Continuous)

Discrete Convolution

$$y[n] = h[n] * x[n]$$

$$y[n] = \sum_{k=0}^{M} h_k x[n-k]$$

FIR Frequency Response

$$H(\hat{\omega}) = \sum_{k=0}^{M} h_k e^{-j\hat{\omega}k}$$

Example sinusoidal sampling

f = 1kHz

$$x(t) = 4\cos(2\pi f t)$$

$$t = nT_s = \frac{n}{f_s} \quad x[n] = 4\cos\left(2\pi\frac{f}{f_s}n\right)$$

$$\hat{\omega} = 2\pi\frac{f}{f_s}$$

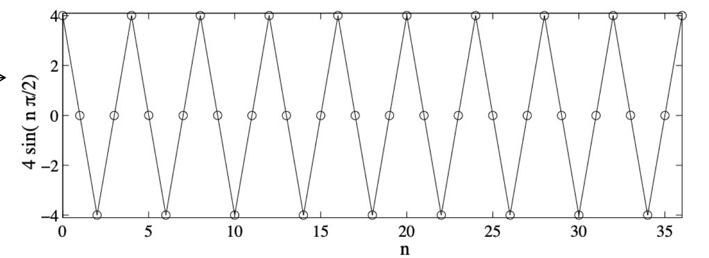
f_s	$\hat{\omega}$
4kHz	$\pi/2$
2kHz	π.
1.333kHz	$3\pi/2 = -\pi/2$
1kHz	2π $(\cos(2n\pi) = 1)$

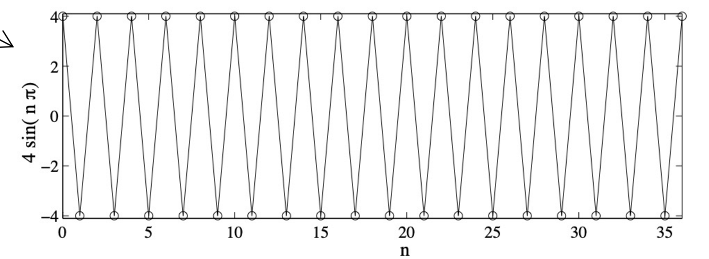
$4\cos(\frac{\pi}{2}n)$

$4\cos(\pi n)$

$4\cos(-\frac{\pi}{2}n)$

4





$$x(t) = 3\cos(2\pi f t) + \cos(6\pi f t)$$

$$x[n] = 3\cos\left(2\pi\frac{f}{f_s}n\right) + \cos\left(6\pi\frac{f}{f_s}n\right)$$

$$\hat{\omega}_1 = 2\pi\frac{f}{f_s}, \hat{\omega}_3 = 6\pi\frac{f}{f_s}$$

f_s	$\hat{\omega}_1$	$\hat{\omega}_2$
18kHz	$\pi/9$.	$\pi/3$
6kHz.	$\pi/3$.	π.
3kHz.	$2\pi/3$	2π $(\cos(2n\pi) = 1)$

$3\cos(\frac{\pi}{9}n) + \cos(\frac{\pi}{3}n)$

$3\cos(\frac{\pi}{3}n) + \cos(\pi n)$

$3\cos(\frac{2\pi}{3}n) + 1$