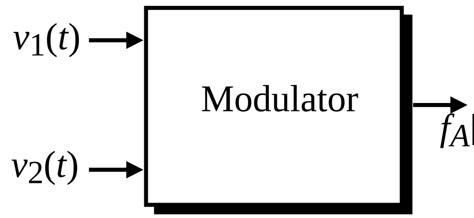
ANALOG MULTIPLIERS

Modulators vs. Multipliers

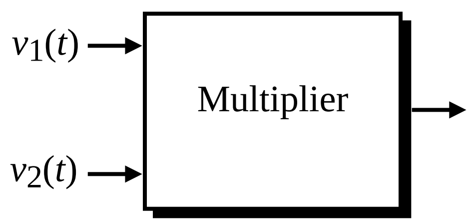
A *modulator* is a circuit with multiple inputs where one input can modify or control the signal flow from another input to the output.



$$v_{out}(t) = f_A[v_1(t)]\cdot f_B[v_2(t)] \qquad (11\text{-}1)$$

where f_A and f_B are two arbitrary functions of $v_1(t)$ and $v_2(t)$, respectively.

A *multiplier* is a modulator where f_A and f_B are linear functions of $v_1(t)$ and $v_2(t)$.

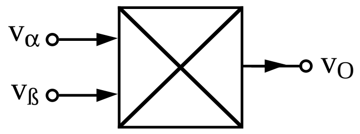


$$v_{out}(t) = K_1 v_1(t)\cdot v_2(t) \qquad (11\text{-}2)$$

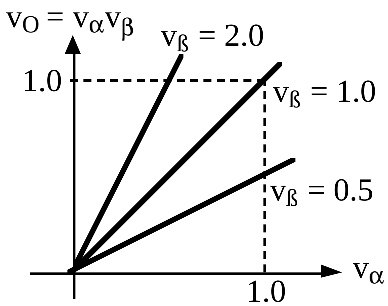
Applications of Multipliers

- Nonlinear analog signal processing
- Mixing
- Modulation and demodulation
- Frequency translation

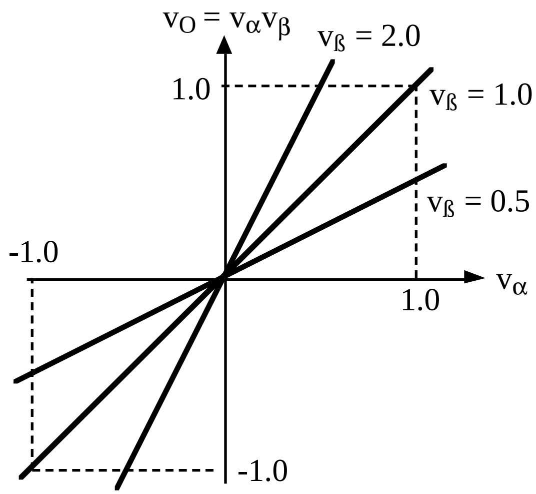
Symbol



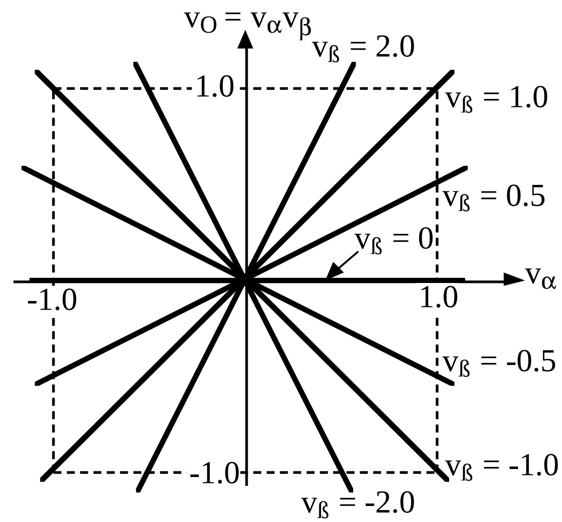
Types of Multiplication



One-Quadrant Multiplier



Two-Quadrant Multiplier



Four-Quadrant Multiplier

Scaling

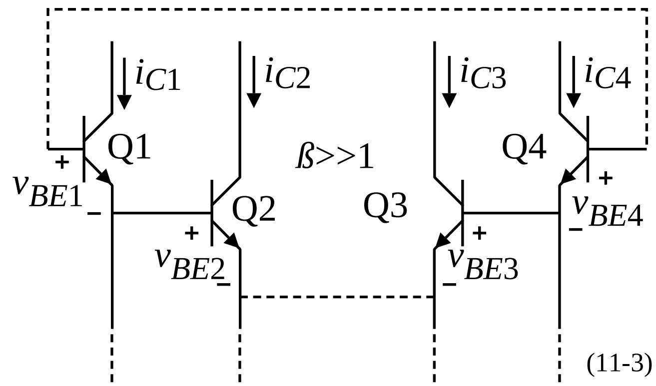
Let K_m be a scaling constant such that

$$v_O = K_m v_\alpha v_\beta.$$

If V_{max} = max $|v_\alpha|$ or max $|v_\beta|$, then $K_m = \dfrac{1}{V_{max}}$ so that max $|v_O| = V_{max}$.

Basic Principle of Analog Bipolar Multiplier - Gilbert Cell

Circuit:



(11-3)

Note that

$$v_{BE1} + v_{BE2} = v_{BE3} + v_{BE4}$$

Substituting for v_{BE} by,

$$v_{BE} = = V_t \, ln\left(\frac{i_C}{I_s}\right)$$

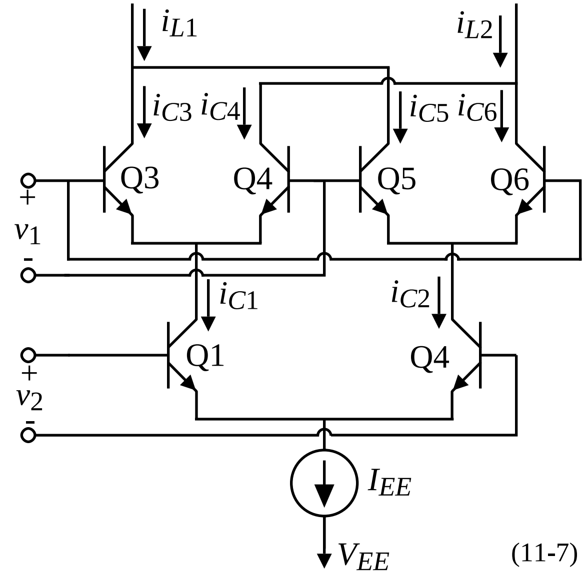
Gives,

$$ln\left(\frac{i_{C1}}{I_{s1}}\right) + ln\left(\frac{i_{C2}}{I_{s2}}\right) = ln\left(\frac{i_{C3}}{I_{s3}}\right) + ln\left(\frac{i_{C4}}{I_{s4}}\right) \quad \Rightarrow \quad ln\left(\frac{i_{C1}i_{C2}}{I_{s1}I_{s2}}\right) = ln\left(\frac{i_{C3}i_{C4}}{I_{s3}I_{s4}}\right) \Rightarrow \frac{i_{C1}i_{C2}}{I_{s1}I_{s2}} = \frac{i_{C3}i_{C4}}{I_{s3}I_{s4}}$$

If Q1 through Q4 are matched, then $I_{s1} = I_{s2} = I_{s3} = I_{s4}$, and

$$\boxed{i_{C1}i_{C2} = i_{C3}i_{C4}}$$

Gilbert Multiplier Cell



(11-7)

Problems with a simple modulator are:

- 2-quadrant, $v_2 > 0$
- Small signal, $v_1 < 2V_t \approx 50\text{mV}$

Solution - Gilbert Cell:

$$i_{C3} = \frac{i_{C1}}{1 + \exp(-v_1/V_t)}, \quad i_{C4} = \frac{i_{C1}}{1 + \exp(v_1/V_t)}$$

$$i_{C5} = \frac{i_{C2}}{1 + \exp(v_1/V_t)}, \quad i_{C6} = \frac{i_{C2}}{1 + \exp(-v_1/V_t)},$$

$$i_{C1} = \frac{I_{EE}}{1 + \exp(-v_2/V_t)} \quad \text{and } i_{C2} = \frac{I_{EE}}{1 + \exp(v_2/V_t)}$$

Let $\Delta i_C = i_{L1} - i_{L2} = (i_{C3} + i_{C5}) - (i_{C4} + i_{C6})$, therefore

$$i_{C3} = \frac{I_{EE}}{[1+\exp(-v_1/V_t)][1+\exp(-v_2/V_t)]}, \qquad i_{C4} = \frac{I_{EE}}{[1+\exp(v_1/V_t)][1+\exp(-v_2/V_t)]}$$

$$i_{C5} = \frac{I_{EE}}{[1+\exp(v_1/V_t)][1+\exp(v_2/V_t)]}, \qquad i_{C6} = \frac{I_{EE}}{[1+\exp(-v_1/V_t)][1+\exp(v_2/V_t)]}$$

Writing Δi_C as $\Delta i_C = (i_{C3} - i_{C6}) + (i_{C5} - i_{C4})$ gives

$$\Delta i_C = I_{EE}\left[\frac{1}{1+\exp(-v_1/V_t)} - \frac{1}{1+\exp(v_1/V_t)}\right]\left[\frac{1}{1+\exp(-v_2,V_t)} - \frac{1}{1+\exp(v_2,V_t)}\right]$$

Note that: $$\frac{1}{1+e^{-x}} - \frac{1}{1+e^{x}} = \frac{e^{x/2}}{e^{x/2} + e^{-x/2}} - \frac{e^{-x/2}}{e^{x/2} + e^{-x/2}} = \frac{e^{x/2} - e^{-x/2}}{e^{x/2} + e^{-x/2}} = \tanh(x/2)$$

Gilbert Multiplier Cell - Continued

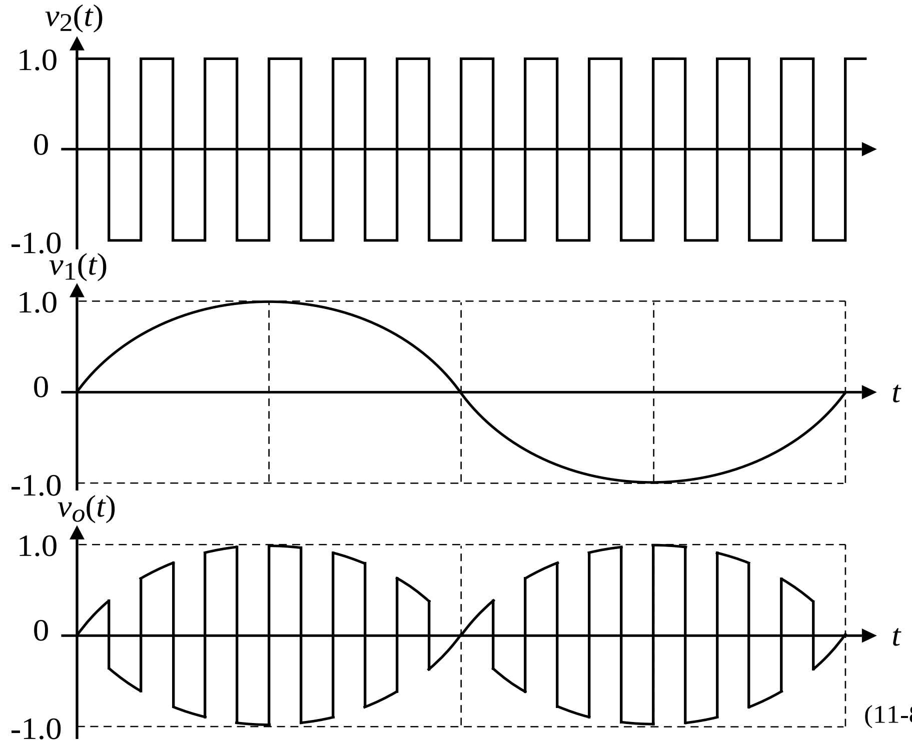
$\therefore\ \Delta i_C = I_{EE}\tanh(v_1/2V_t)\tanh(v_2/2V_t)$ which solves the two-quadrant problem.

Assume that, $v_{OUT} = R(i_{L1} - i_{L2}) = R\Delta i_C = RI_{EE}\tanh(v_1/2V_t)\tanh(v_2/2V_t)$

Synchronous Modulator:

$v_1 \ll 2V_t$ and $v_2 \gg 2V_t \Rightarrow v_{OUT} \approx RI_{EE}v_2\tanh(v_1/2V_t)$

Waveforms:

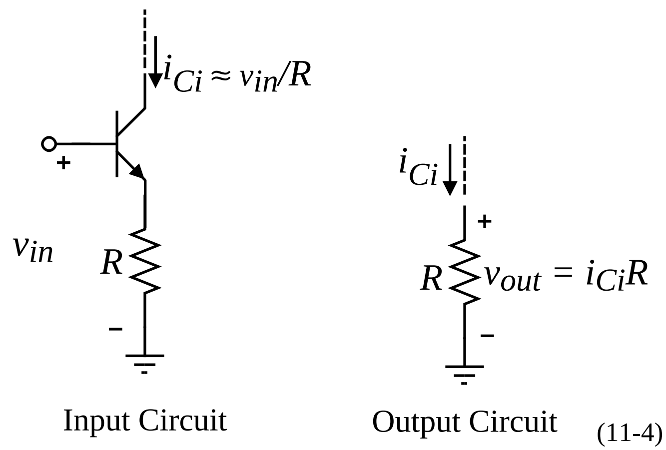


(11-8)

Problem: $v_1 < 2V_t$ and $v_2 < 2V_t$

Using Voltages for the Gilbert Cell

Voltage Equivalent:



(11-4)

Therefore the previouse results can be converted to,

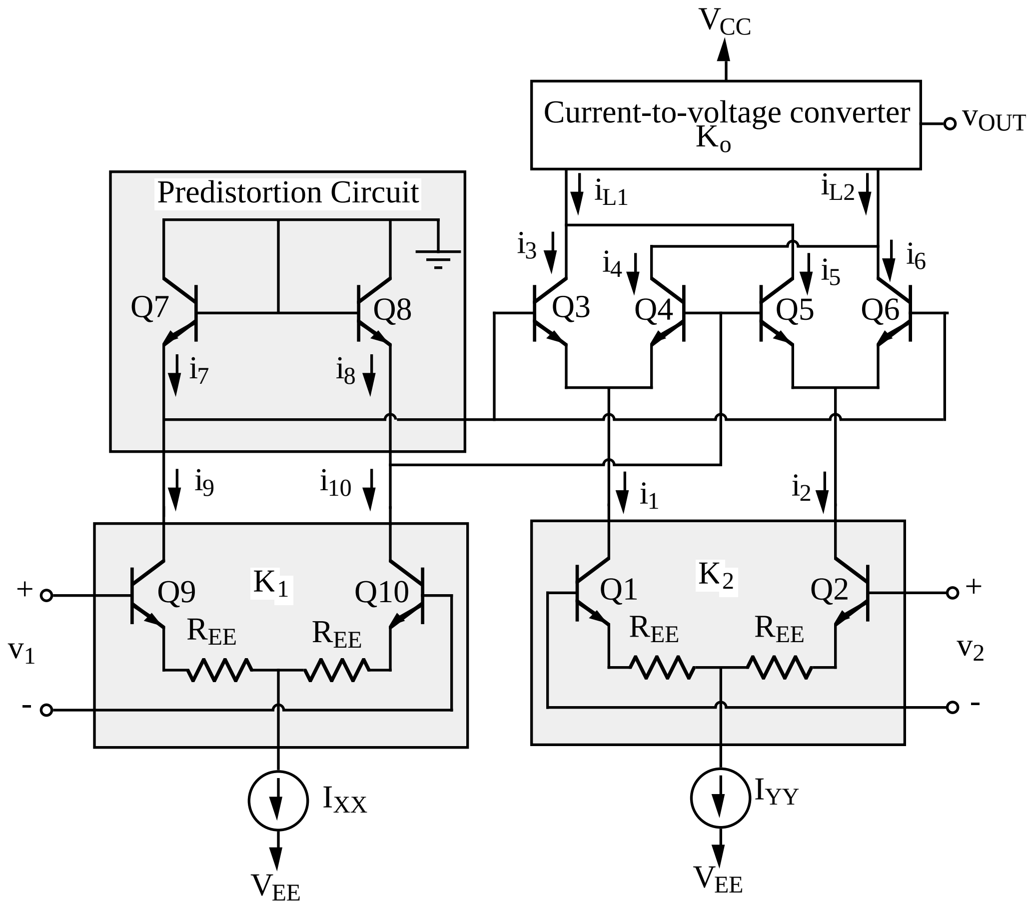
$$v_{in}v_2 = v_3v_4$$

Let v_4 be an output, then

$$v_4 = v_{out} = \frac{v_{in}v_2}{v_3}$$

Four Quadrant Linear Multiplier

Circuit:



Four Quadrant Linear Multiplier - Continued

Analysis

Note that, $v_{BE3} - v_{BE4} + v_{BE5} - v_{BE6} = 0 \rightarrow V_T\ln\left(\frac{i_3}{I_{s3}}\right) - V_T\ln\left(\frac{i_4}{I_{s4}}\right) + V_T\ln\left(\frac{i_5}{I_{s5}}\right) - V_T\ln\left(\frac{i_6}{I_{s6}}\right) = 0$

Assuming matched transistors gives $i_3 i_5 = i_4 i_6$, $i_9 i_3 = i_4 i_{10}$ and $i_9 i_6 = i_5 i_{10}$

Also note that, $i_1 = i_3 + i_4$, $i_2 = i_5 + i_6$, $i_{L1} = i_3 + i_5$, $i_{L2} = i_4 + i_6$, and $I_{XX} = i_9 + i_{10}$

Assume that $i_9 - i_{10} = \frac{v_1}{K_1}$, $i_1 - i_2 = \frac{v_2}{K_2}$, and $v_{OUT} = K_o(i_{L1} - i_{L2})$ where $K_1 = K_2 \approx 2R_{EE}$

Now,

$$v_{OUT}=K_o[(i_4+i_6)-(i_3+i_5)] = K_o\left[\left(i_4+i_5\frac{i_{10}}{i_9}\right)-\left(i_4\frac{i_{10}}{i_9}+i_5\right)\right] = K_o\left[i_4-i_4\frac{i_{10}}{i_9}-i_5+i_5\frac{i_{10}}{i_9}\right] = K_o(i_4-i_5)\left(1-\frac{i_{10}}{i_9}\right) = K_o\left(\frac{i_9-i_{10}}{i_9}\right)(i_4-i_5)$$

Next, find $i_4 - i_5$ in terms of $i_1 - i_2$ as follows.

$$i_1 - i_2 = (i_3 + i_4) - (i_5 + i_6) = \left(i_4\frac{i_{10}}{i_9} + i_4\right) - \left(i_5 + i_5\frac{i_{10}}{i_9}\right) = \left(\frac{i_9 + i_{10}}{i_9}\right)(i_4 - i_5)$$

Therefore, $i_4 - i_5 = \left(\frac{i_9}{i_9 + i_{10}}\right)(i_1 - i_2)$ which is the desired result.

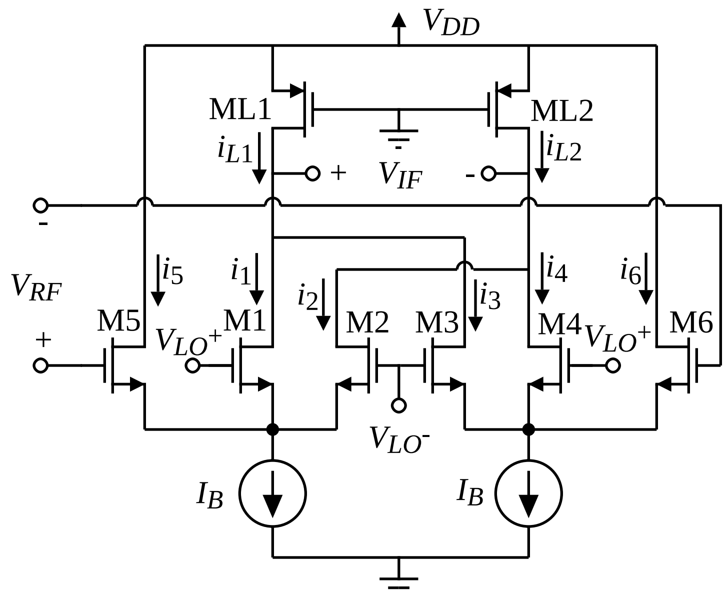
Substituting gives

$$v_{OUT} = K_o\left(\frac{i_9 - i_{10}}{i_9}\right)\left(\frac{i_9}{i_9 + i_{10}}\right)(i_1 - i_2) = K_o\frac{(i_9 - i_{10})(i_1 - i_2)}{i_9 + I_{10}} = \frac{K_o}{I_x}(I_9 - i_{10})(i_1 - i_2)$$

$$\therefore \quad \boxed{v_{OUT}= \frac{K_o}{I_{XX}K_1K_2} v_1 v_2 = K_m v_1 v_2}$$

2V, High-Frequency CMOS Multiplier†

Based on the Gilbert cell with two source-followers as current modulators.



Comparison with the Gilbert cell:

- Can operate at a lower supply voltage because the mixer does not use stacking
- Source followers give better linearity
- Has a smaller mixer gain because sharing the bias currents with the followers reduces g_m

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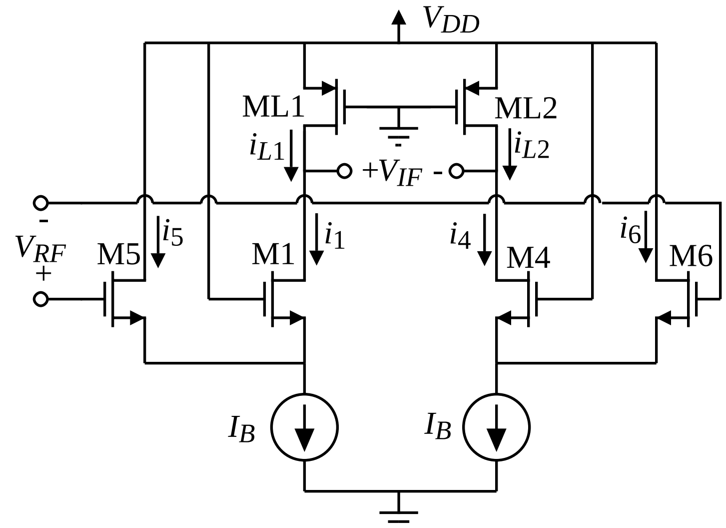
† K-K Kan, D. Ma, K-C Mak and H.C. Luong, "Design Theory and Performance of a 1-GHz CMOS Downconversion and Upconversion Mixers," *Analog Integrated Circuit and Signal Processing*, Vol. 24, No. 2, pp. 101-111, July 2000.

How Does the Previous Multiplier Work?

The local oscillator is sufficiently large that M1-M4 are fully switched on or off. Therefore, the mixer can be redrawn below and consists of two pairs of unbalanced source coupled MOSFETs.



The IF output can be found as follows:

The differential drain current of an unbalanced source-coupled pair can be written as

$$\Delta i_D = i_{D1} - i_{D5} = I_{DC}+i_{SQ} + i_{non}$$

where

$$I_{DC} = \frac{(W_1/L1)-(W_5/L_5)}{(W_1/L1)+(W_5/L_5)} I_B$$

which is the output offset current due to the asymmetry of the unbalanced source-coupled pair.

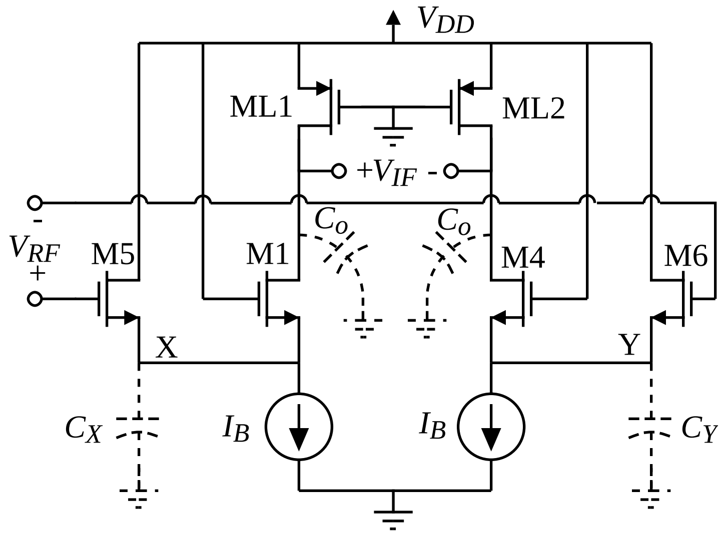
$$i_{SQ} = \frac{(W_1/L1)(W_5/L_5)[(W_5/L_5)-(W_1/L_1)]K_n'}{[(W_1/L1)+(W_5/L_5)]^2} V_i^{\,2}$$

which is a current proportional to the square of the differential input voltage, $V_i = V_{G1} - V_{G5}$ and where VG is the gate voltage derived from the inherent square law model

$$i_{non} = \frac{2(W_1/L_1)(W_5/L_5)K_n'V_i}{[(W_1/L_1)+(W_5/L_5)]^2} \sqrt{[(W_1/L_1)+ (W_5/L_5)]\frac{2I_B}{K_n'} - (W_1/L_1)\,(W_5/L_5)Vi^2}$$

which is the portion of current that causes harmonic distortion.

Frequency Response of the Previous Multiplier



There are four poles and one zero.

Dominant pole:

$$p_1 = \frac{1}{R_o C_o}$$

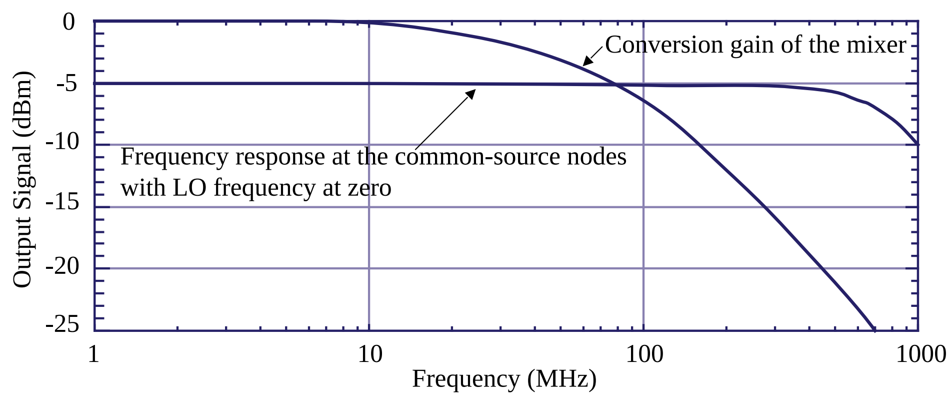
where R_o is the output resistance at V_{IF}.

Second pole:

$$p_2 = \frac{g_{m1}+g_{mbs1}+g_{m5}+g_{mbs5}+g_{o5}+g_{o7}}{C_x+C_{gs1}+C_{gs5}}$$

where g_{o7} is the output conductance of I_B and C_x is the capacitance contributed by the biasing transistor as well as the cutoff transistor. The other two poles and zero are higher in magnitude and can be neglected.

Frequency response:



Four-Quadrant CMOS Multiplier†

Small Signal Analysis:

$i_{out} = i_7 - i_8 = (i_3+i_5) - (i_4+i_6)$

$= (i_3 - i_4) - (i_6 - i_5)$

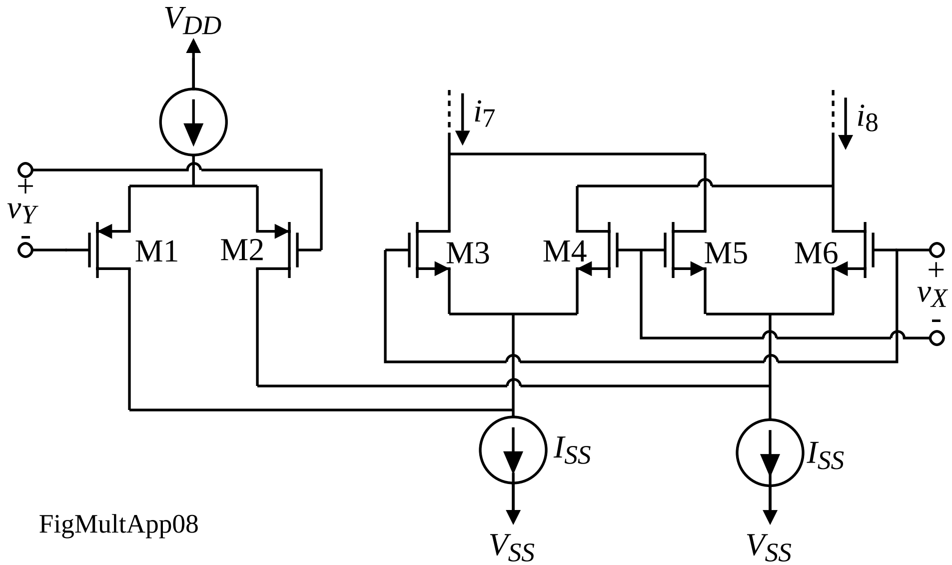
where

$i_3 = \frac{g_{m34}v_x}{2}$, $i_4 = \frac{-g_{m34}v_x}{2}$,

$i_6 = \frac{g_{m56}v_x}{2}$, and $i_5 = \frac{-g_{m56}v_x}{2}$

$\therefore\ i_{out} = g_{m34}v_x - g_{m56}v_x$

$= \left(\sqrt{2K_NS_NI_{34}} - \sqrt{2K_NS_NI_{56}}\right) v_x$



FigMultApp08

However, $I_{34} = I_{SS} - (0.5I_{DD}+i_1)$ and $I_{56} = I_{SS} - (0.5I_{DD}+i_2)$

If $I_{DD} = I_{SS}$, then $I_{34} = 0.5I_{SS} - i_1$ and $I_{56} = 0.5I_{SS} - i_2$

$\therefore\ i_{out} = \sqrt{2K_NS_N}\left(\sqrt{I_{34}} - \sqrt{I_{56}}\right)v_x = \sqrt{2K_NS_N}\left(\sqrt{0.5I_{SS}-i_1} - \sqrt{0.5I_{SS} - i_2}\right)v_x$

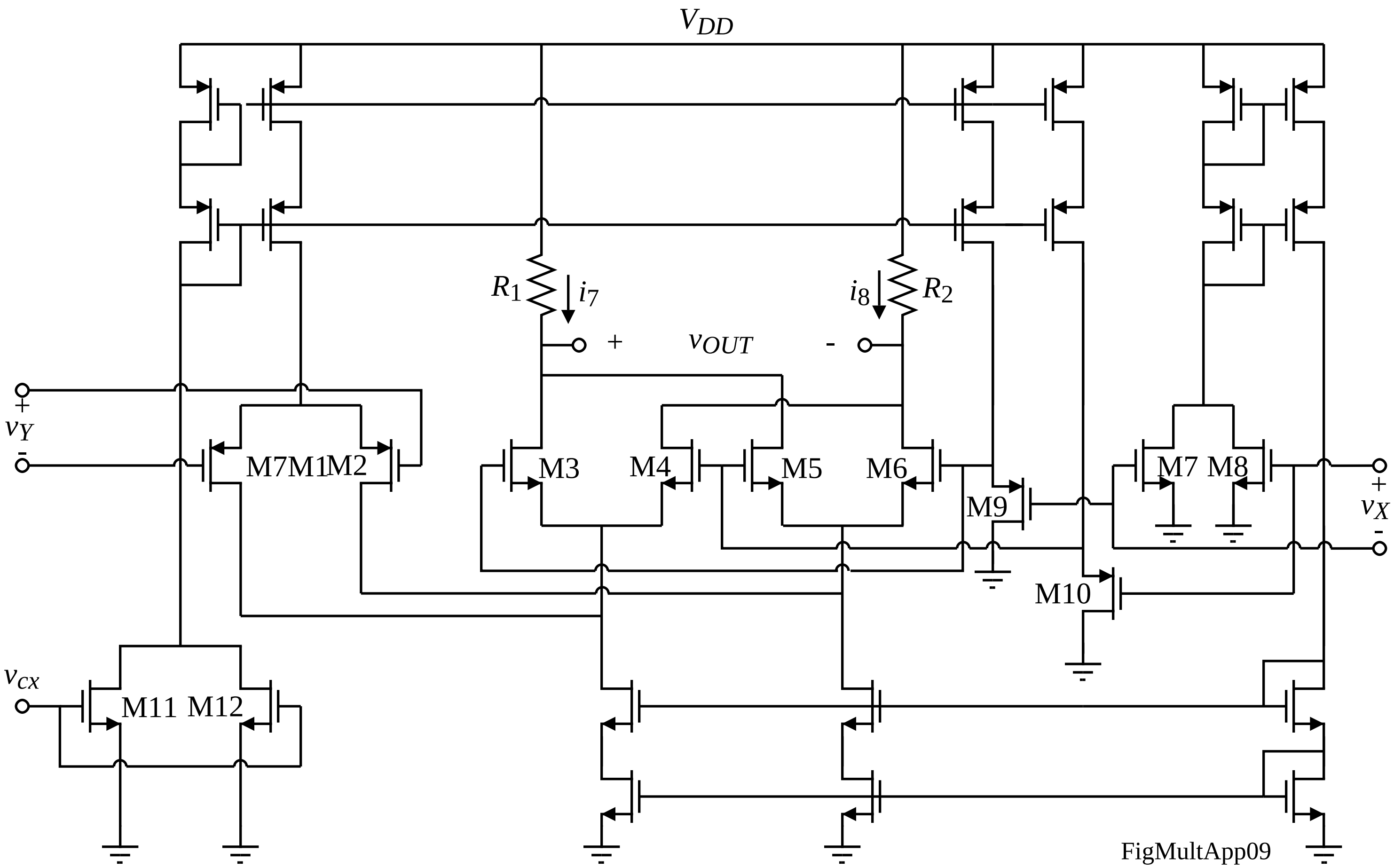
$= \sqrt{K_NS_NI_{SS}}\left(\sqrt{1 -(2i_1/I_{SS}\text{-})} - \sqrt{1 -(2i_2/I_{SS}\text{-})}\right)v_x \approx \sqrt{K_NS_NI_{SS}}\,[(-i_1/I_{SS}\text{-}) + (i_2/I_{SS}\text{-})]v_x$

$i_{out} = \sqrt{\frac{K_NS_N}{I_{SS}}}\,(-i_1+i_2)\,v_x = \sqrt{\frac{K_NS_N}{I_{SS}}}\left(\frac{g_{m12}v_y}{2} + \frac{g_{m12}v_y}{2}\right)v_x = \sqrt{\frac{K_NS_N}{I_{SS}}}\,g_{m12}v_yv_x = \sqrt{2K_NS_NK_PS_P}\,v_xv_y$

† Babanezhad and Temes - *JSSC*, Dec. 1985.

CMOS Four-Quadrant Multiplier

Complete circuit:



v_{CX} is a voltage used to establish the common mode in the multiplier.

Four-Quadrant CMOS Multiplier

Zero-IF Downconverters:

Uses FETs in the triode region to achieve a linear four-quadrant multiplier.

Ideal Operation (Op Amp Gain = ∞ and $v_i = 0$):

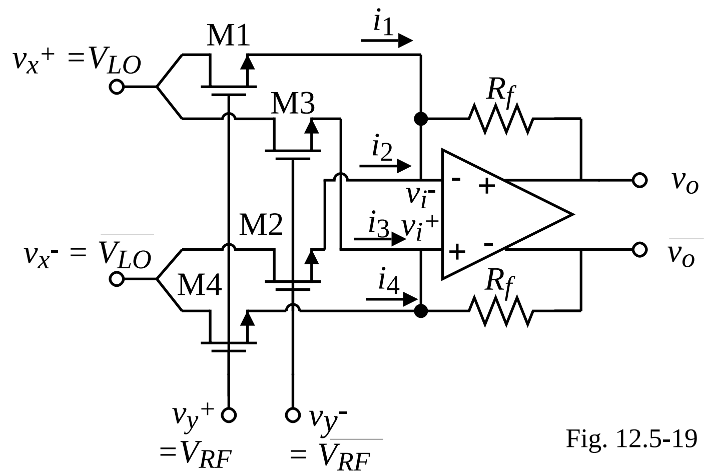


Fig. 12.5-19

$$i_1 = K'\left[\left(V_{GS} + \frac{v_y}{2} - V_T\right)\frac{v_x}{2} - \frac{1}{2}\left(\frac{v_x}{2}\right)^2\right]$$

$$i_2 = K'\left[\left(V_{GS} - \frac{v_y}{2} - V_T\right)\left(\frac{-v_x}{2}\right) - \frac{1}{2}\left(\frac{-v_x}{2}\right)^2\right]$$

$$i_3 = K'\left[\left(V_{GS} - \frac{v_y}{2} - V_T\right)\frac{v_x}{2} - \frac{1}{2}\left(\frac{v_x}{2}\right)^2\right]$$

$$i_4 = K'\left[\left(V_{GS} + \frac{v_y}{2} - V_T\right)\left(\frac{-v_x}{2}\right) - \frac{1}{2}\left(\frac{-v_x}{2}\right)^2\right]$$

$$v_o = R(v_o^+ - v_o^-) = RK'(-i_4 - i_3 + i_1 + i_2) = RK'\left(\frac{v_x v_y}{2} + \frac{v_x v_y}{2}\right) = RK' v_x v_y$$

∴ $\boxed{v_o = RK' v_{LO} v_{RF^\circ} = G_T v_{LO} v_{RF}}$

where the gain, $G_T = RK'$

Four-Quadrant CMOS Multiplier - Continued

Previous circuit with a finite op amp gain (A):

$$v_o = v_o^+ - v_o^- = A\,(v_i^+ - v_i^-) \;\Rightarrow\quad v_i^+ = \frac{v_o}{2A} \quad \text{and} \quad v_i^- = \frac{-v_o}{2A}$$

$$i_1 = K'\left[\left(V_{GS} + \frac{v_y}{2} - \frac{v_o}{2A} - V_T\right)\left(\frac{v_x}{2} - \frac{v_o}{2A}\right) - \frac{1}{2}\left(\frac{v_x}{2} - \frac{v_o}{2A}\right)^2\right]$$

$$i_2 = K'\left[\left(V_{GS} - \frac{v_y}{2} - \frac{v_o}{2A} - V_T\right)\left(-\frac{v_x}{2} - \frac{v_o}{2A}\right) - \frac{1}{2}\left(\frac{v_x}{2} + \frac{v_o}{2A}\right)^2\right]$$

$$i_3 = K'\left[\left(V_{GS} - \frac{v_y}{2} + \frac{v_o}{2A} - V_T\right)\left(\frac{v_x}{2} + \frac{v_o}{2A}\right) - \frac{1}{2}\left(\frac{v_x}{2} + \frac{v_o}{2A}\right)^2\right]$$

$$i_4 = K'\left[\left(V_{GS} + \frac{v_y}{2} + \frac{v_o}{2A} - V_T\right)\left(-\frac{v_x}{2} + \frac{v_o}{2A}\right) - \frac{1}{2}\left(-\frac{v_x}{2} + \frac{v_o}{2A}\right)^2\right]$$

$$v_o = \frac{R}{1 + \frac{1}{A}}(-i_4 - i_3 + i_1 + i_2) = \frac{RK'}{1 + \frac{1}{A}}\left[\frac{v_x v_y}{2} + \frac{V_T v_o}{A} - \frac{V_{GS} v_o}{A} + \frac{v_x v_y}{2} - \frac{V_{GS} v_o}{A} + \frac{V_T v_o}{A}\right]$$

$$v_o\left[\left(1 + \frac{1}{A}\right)\frac{2RK'}{A}(V_{GS} - V_T)\right] = RK' v_x v_y$$

$$\therefore \quad \boxed{v_o = \frac{G_T}{1 + \frac{1}{A} + \frac{2G_T}{A}(V_{GS} - V_T)}\, v_x v_y}$$

Four-Quadrant CMOS Analog Multiplier - Continued

Previous circuit with a finite op amp gain (A) and a threshold variation (ΔV_T^+ and ΔV_T^-):

$$i_1 = K'\left[\left(V_{GS} + \frac{v_y}{2} - \frac{v_o}{2A} - V_T - \Delta V_T^+\right)\left(\frac{v_x}{2} - \frac{v_o}{2A}\right) - \frac{1}{2}\left(\frac{v_x}{2} - \frac{v_o}{2A}\right)^2\right]$$

$$i_2 = K'\left[\left(V_{GS} - \frac{v_y}{2} - \frac{v_o}{2A} - V_T - \Delta V_T^+\right)\left(-\frac{v_x}{2} - \frac{v_o}{2A}\right) - \frac{1}{2}\left(\frac{v_x}{2} + \frac{v_o}{2A}\right)^2\right]$$

$$i_3 = K'\left[\left(V_{GS} - \frac{v_y}{2} + \frac{v_o}{2A} - V_T - \Delta V_T^-\right)\left(\frac{v_x}{2} + \frac{v_o}{2A}\right) - \frac{1}{2}\left(\frac{v_x}{2} + \frac{v_o}{2A}\right)^2\right]$$

$$i_4 = K'\left[\left(V_{GS} + \frac{v_y}{2} + \frac{v_o}{2A} - V_T - \Delta V_T^-\right)\left(-\frac{v_x}{2} + \frac{v_o}{2A}\right) - \frac{1}{2}\left(-\frac{v_x}{2} + \frac{v_o}{2A}\right)^2\right]$$

$$v_o = \frac{R}{1 + \frac{1}{A}}(-i_4 - i_3 + i_1 + i_2)$$

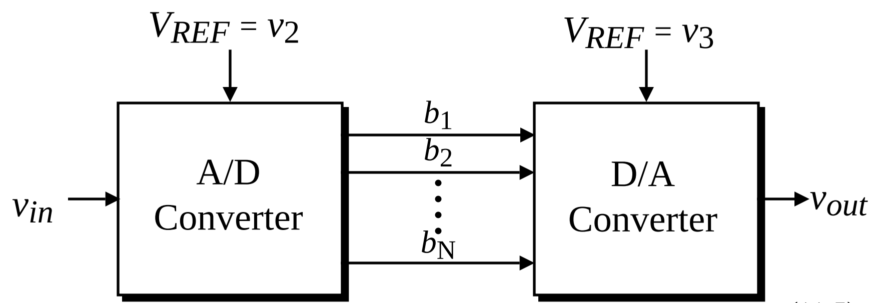
$$= \frac{RK'}{1 + \frac{1}{A}}\left[\frac{v_xv_y}{2} + \frac{\Delta V_T^+ v_o}{A} + \frac{V_Tv_o}{A} + \frac{v_o^2}{2A^2} - \frac{V_{GS}v_o}{A} + \frac{v_xv_y}{2} + \frac{\Delta V_T^- v_o}{A} + \frac{V_Tv_o}{A} - \frac{v_o^2}{2A^2} - \frac{V_{GS}v_o}{A}\right]$$

$$v_o = \frac{G_T}{1 + \frac{1}{A}}\left[v_xv_y - \frac{2v_o}{A}(V_{GS} - V_T) + \frac{v_o}{A}(\Delta V_T^+ + \Delta V_T^-)\right]$$

$$\therefore \quad \boxed{v_o = \frac{G_T}{\left(1 + \frac{1}{A}\right)\left[1 + \frac{2G_T}{1 + A}(V_{GS} - V_T) - \frac{G_T}{1 + A}(\Delta V_T^+ + \Delta V_T^-)\right]} v_xv_y}$$

An Analog Multiplier using a A/D and a D/A Converter

Circuit:



(11-5)

The digital word converted by the ADC is,

$$D = \left[\frac{b_1}{2} + \frac{b_2}{4} + \frac{b_3}{8} + \cdots + \frac{b_N}{2^{N-1}}\right]$$

The input of the A/D converter can be expressed as

$$v_{in} = V_{REF}\left[\frac{b_1}{2} + \frac{b_2}{4} + \frac{b_3}{8} + \cdots + \frac{b_N}{2^{N-1}}\right] = v_2 D \qquad \Rightarrow \qquad D = \frac{v_{in}}{v_2}$$

The DAC does the following,

$$\boxed{v_{out} = D\, v_3 = \frac{v_{in} v_3}{v_2}}$$

A multiplier with N-bits of precision.

Applications of Multipliers

1.) Frequency Doubling

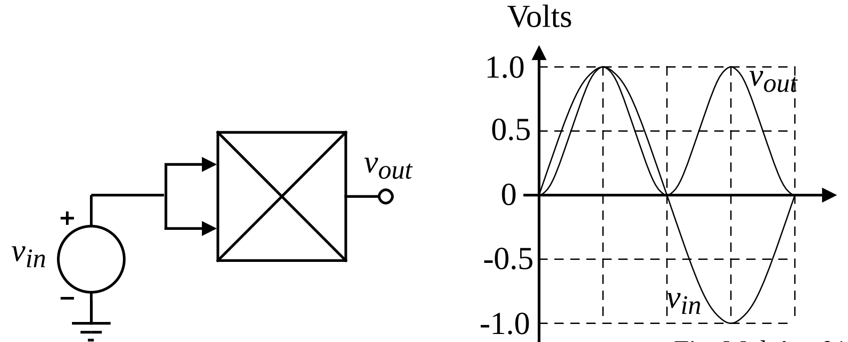


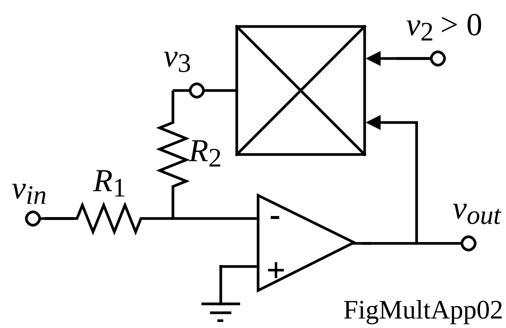
Fig. MultApp01

2.) Division (positive divisor)

$$\frac{v_{in}}{R_1} = -\frac{v_3}{R_2} \Rightarrow v_3 = -\frac{R_2}{R_1} v_{in}$$

$$v_3 = v_{out} \cdot v_{in}$$

$$\therefore \quad v_{out} \cdot v_2 = -\frac{R_2}{R_1} v_{in}$$

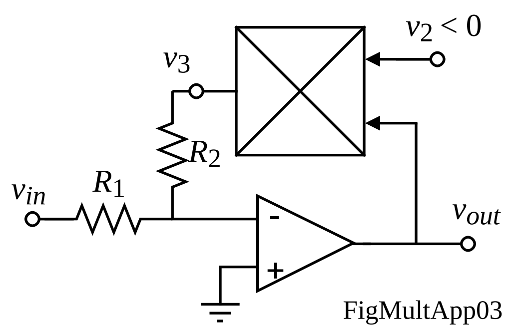


FigMultApp02

$$\boxed{v_{out} = -\frac{R_2}{R_1}\frac{v_{in}}{v_2}, \; v_2 > 0}$$

3.) Division (negative divisor)

$$\boxed{v_{out} = \frac{R_2}{R_1}\frac{v_{in}}{v_2}, \; v_2 < 0}$$

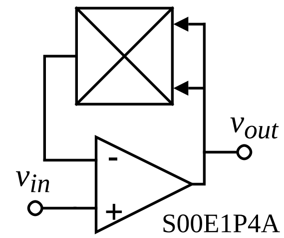


FigMultApp03

Applications - Continued

4.) Square root (v_{in}>0)

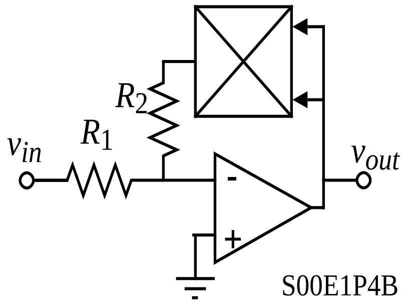
$$v_{in} = K_m v_{out}^2 \quad \Rightarrow \quad \boxed{v_{out} = \sqrt{\frac{v_{in}}{K_m}},\ v_{in} > 0}$$



S00E1P4A

5.) Square root (v_{in}<0)

$$\frac{K_m v_{out}^2}{R_2} + \frac{v_{in}}{R_1} = 0 \Rightarrow \quad \boxed{v_{out} = \sqrt{-\frac{R_2 v_{in}}{K_m R_1}},\ v_{in} < 0}$$

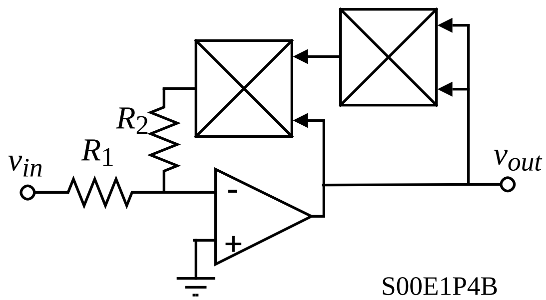


S00E1P4B

6.) Cube root (v_{in}<0)

$$\frac{K_m^2 v_{out}^3}{R_2} + \frac{v_{in}}{R_1} = 0$$

$$\Rightarrow \quad \boxed{v_{out} = \left(-\frac{R_2 v_{in}}{K_m^2 R_1}\right)^{1/3}, v_{in} < 0}$$



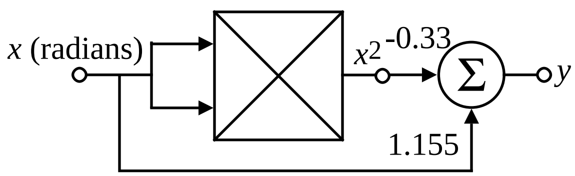
S00E1P4B

Use of Analog Multipliers to Approximate sin(x)

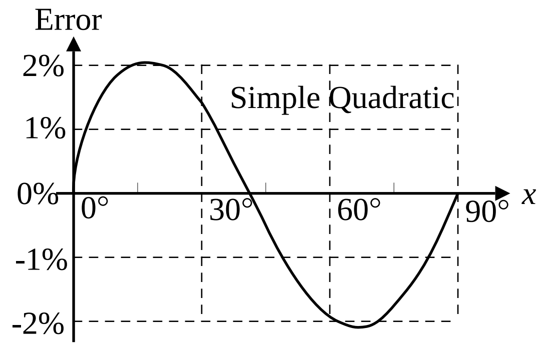
Consider the following approximations to sin(x):

1. Quadratic, one-quadrant, single-multiplier.

   A. Explicit function: $y = 1.155x - 0.33x^2$

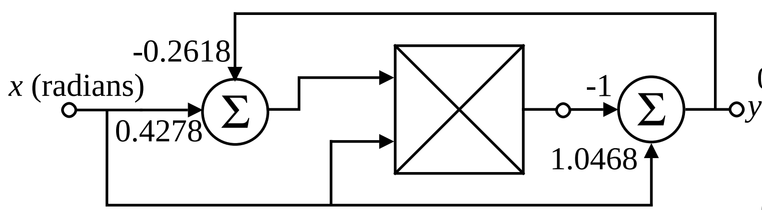


FigMultApp04

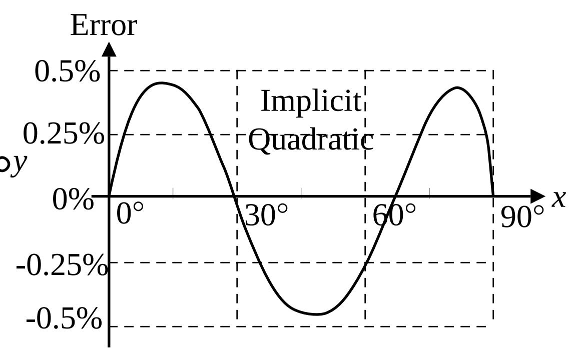


   B. Implicit function: $y = 1.0468x - x(0.4278x - 0.2618y)$

$$y = \frac{1.0468x - 0.4278x^2}{1 - 0.2618x}$$



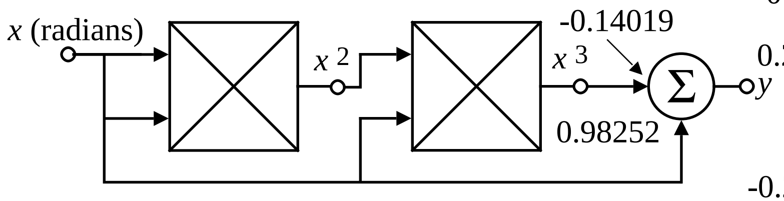
FigMultApp05



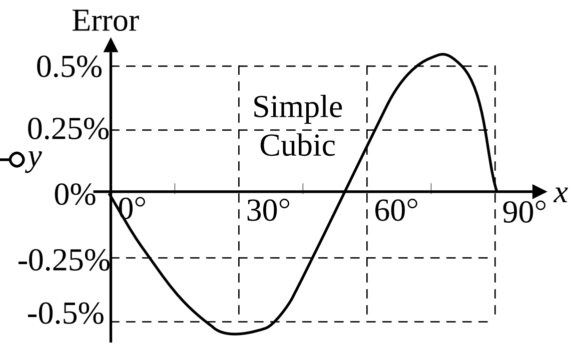
Use of Analog Multipliers to Approximate sin(x) - Continued

2. Cubic, two-quadrant, 2-multiplier

A. Explicit function: $y = 0.98252x - 0.14019x^3$

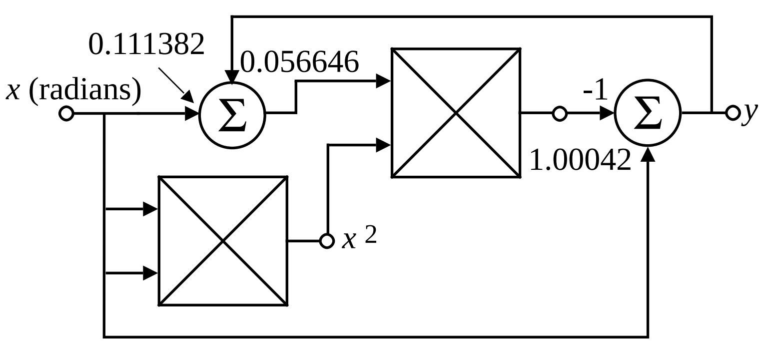


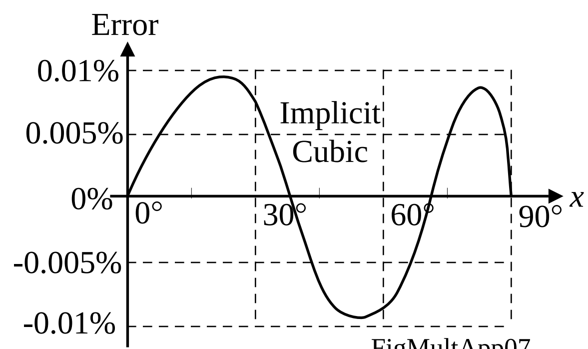
FigMultApp06



B. Implicit function: $y = 1.00042x - x^2(0.111382x + 0.056646y)$

$$y = \frac{1.00042x - 0.111382x^2}{1 + 0.056646x}$$





FigMultApp07