How to handle general linear differential equations with arbitrary inputs (e.g. signals)?

$$\frac{d^4y}{dt^4} + 3\frac{d^3y}{dt^3} + 7\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 25y = x_{in}(t)$$

Change d/dt → s (Heavyside)

$$s^4y + 3s^3y + 7s^2y + 4sy + 25y = x_{in}(t)$$

and solve in some different ways

Laplace Transform (should be Heavyside Transform)

$$F(s) = \int_0^\infty f(t)e^{-st}dt$$

$$s = \sigma + j\omega$$ Complex exponential Basis functions

$$f(t) \rightarrow F(s) \qquad F(s) \rightarrow f(t)$$

Laplace Transform Table

Time Domain (t)	Laplace Domain (s)
$\frac{df(t)}{dt}$	$sF(s)$
$\int f(t)dt$	$\frac{1}{s}F(s)$
$u(t)$	$\frac{1}{s}$
$e^{-at}u(t)$	$\frac{1}{s+a}$
$\sin(\omega_1 t)\, u(t)$	$\frac{\omega_1}{s^2+\omega_1^2}$
$\cos(\omega_1 t)\, u(t)$	$\frac{s}{s^2+\omega_1^2}$
$f(t \rightarrow \infty)$	$\lim_{t\rightarrow 0} sF(s)$
$f(t \rightarrow 0)$	$\lim_{t\rightarrow \infty} sF(s)$

$$\tau^2 \frac{d^2V}{dt^2} + 3\tau \frac{dV}{dt} + 2V(t) = V_{in}(t)$$

$$s^2\tau^2 V(s) + 3s\tau V(s) + 2V(s) = V_{in}(s)$$

$$V(s) = \frac{1}{s^2\tau^2 + 3s\tau + 2} V_{in}(s)$$

$$V(s) = \frac{1}{(s\tau + 2)(s\tau + 1)} V_{in}(s)$$

$$V_{in}(t) = u(t) \rightarrow V_{in}(s) = \frac{1}{s}$$

$$V(s) = \frac{1}{s\tau + 2} \frac{1}{s\tau + 1} \frac{1}{s}$$

$$V(s) = A\frac{\tau}{s\tau + 2} + B\frac{\tau}{s\tau + 1} + C\frac{1}{s}$$

$$V(s) = \frac{1}{2}\frac{1}{s + 2/\tau} - \frac{1}{s + 1/\tau} + \frac{1}{2}\frac{1}{s}$$

$$V_{out}(t) = \left(\frac{1}{2}\left(1 + e^{-2t/\tau}\right) - e^{-t/\tau}\right) u(t)$$

$$\frac{V(s)}{V_{in}(s)} = \frac{1}{s^2\tau^2 + 3s\tau + 2}$$

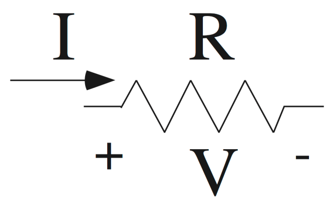
$$V_{in}(t) = \sin(\omega t)$$

$$s = \sigma + j\omega \rightarrow j\omega$$

$$\frac{V(j\omega)}{V_{in}(j\omega)} = \frac{1}{(j\omega)^2\tau^2 + 3(j\omega)\tau + 2}$$

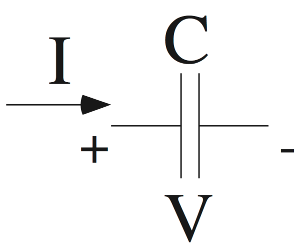
$$V_{in}(t) = \frac{1}{(2 - \omega^2\tau^2)^2 + 9\omega^2\tau^2} \sin\left(\omega t - \tan^{-1}\left(\frac{3\omega\tau}{2 - \omega^2\tau^2}\right)\right)$$

Laplace form of Circuit elements



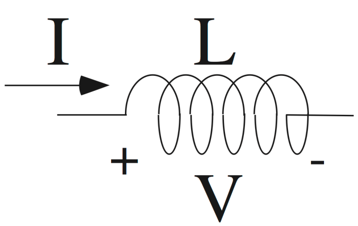
Resistor

$$V = RI \longrightarrow V(s) = R\, I(s)$$



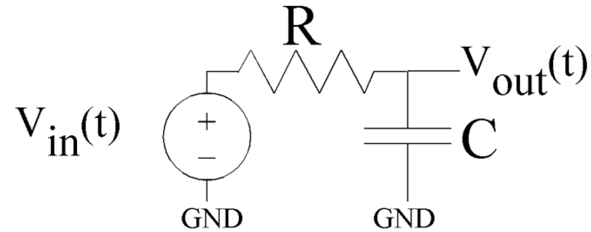
Capacitor

$$I = C\frac{dV}{dt} \longrightarrow I(s) = s\, C\, V(s)$$



Inductor

$$V = L\frac{dI}{dt} \longrightarrow V(s) = s\, L\, I(s)$$



Voltage Divider:

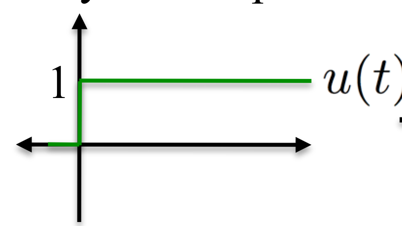
$$V_{out}(s) = V_{in}(s)\frac{1/sC}{R + 1/sC}$$

$$= V_{in}(s)\frac{1}{1 + sRC}$$

$$= V_{in}(s)\frac{1}{1 + s\tau}$$

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{1 + s\tau}$$

Heavyside Step Function



$$u(t) \longrightarrow \frac{1}{s}$$

$$V_{out}(t) = (1 - e^{-t/\tau})u(t)$$

Laplace Transform → Fourier Transform

$$F(s) = \int_0^{\infty} f(t)e^{-st}dt \qquad F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

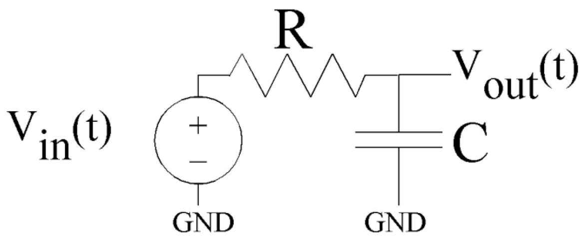
$$s = \sigma + j\omega \qquad \sigma = 0, s \rightarrow j\omega \quad j = \sqrt{-1}$$

$$f(t) \rightarrow F(s) \quad F(s) \rightarrow f(t)$$

Transform: reversible

Quantum Physics: Momentum relates to Fourier Transform of position

1st order example

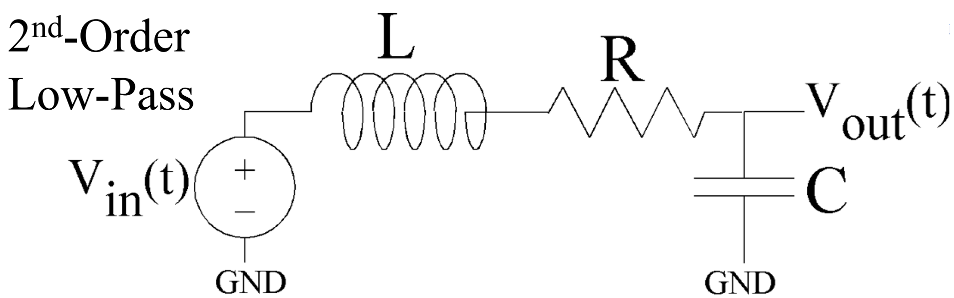


$$s \rightarrow j\omega \qquad \omega = 2\pi f$$

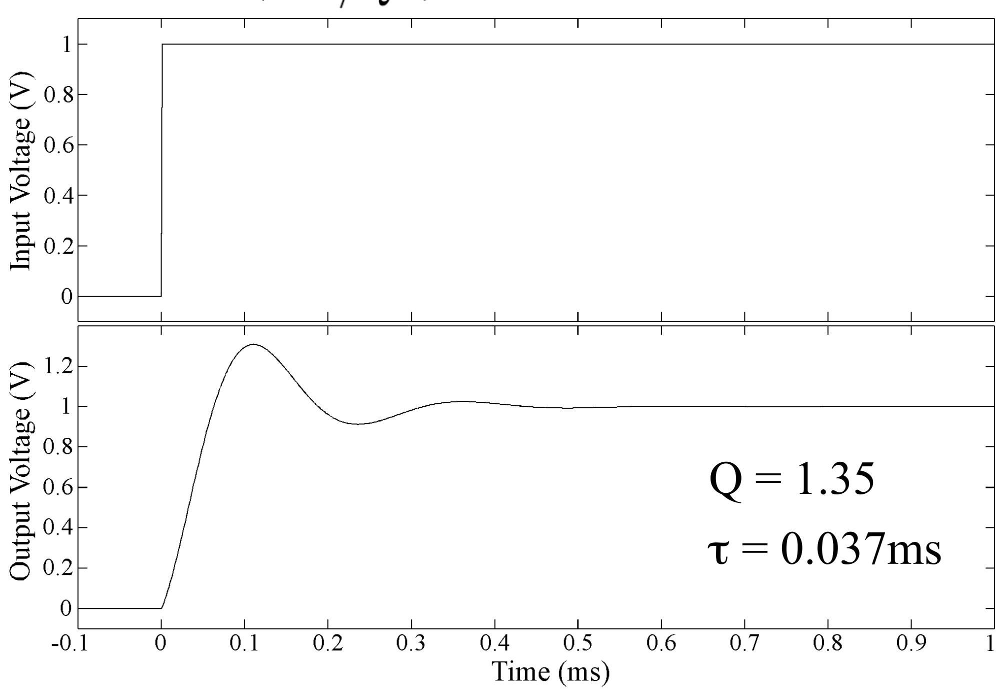
$$H(j\omega) = \frac{V_{out}(j\omega)}{V_{in}(j\omega)} = \frac{1}{1 + j\omega\tau} \qquad |a + jb| = \sqrt{a^2 + b^2}$$

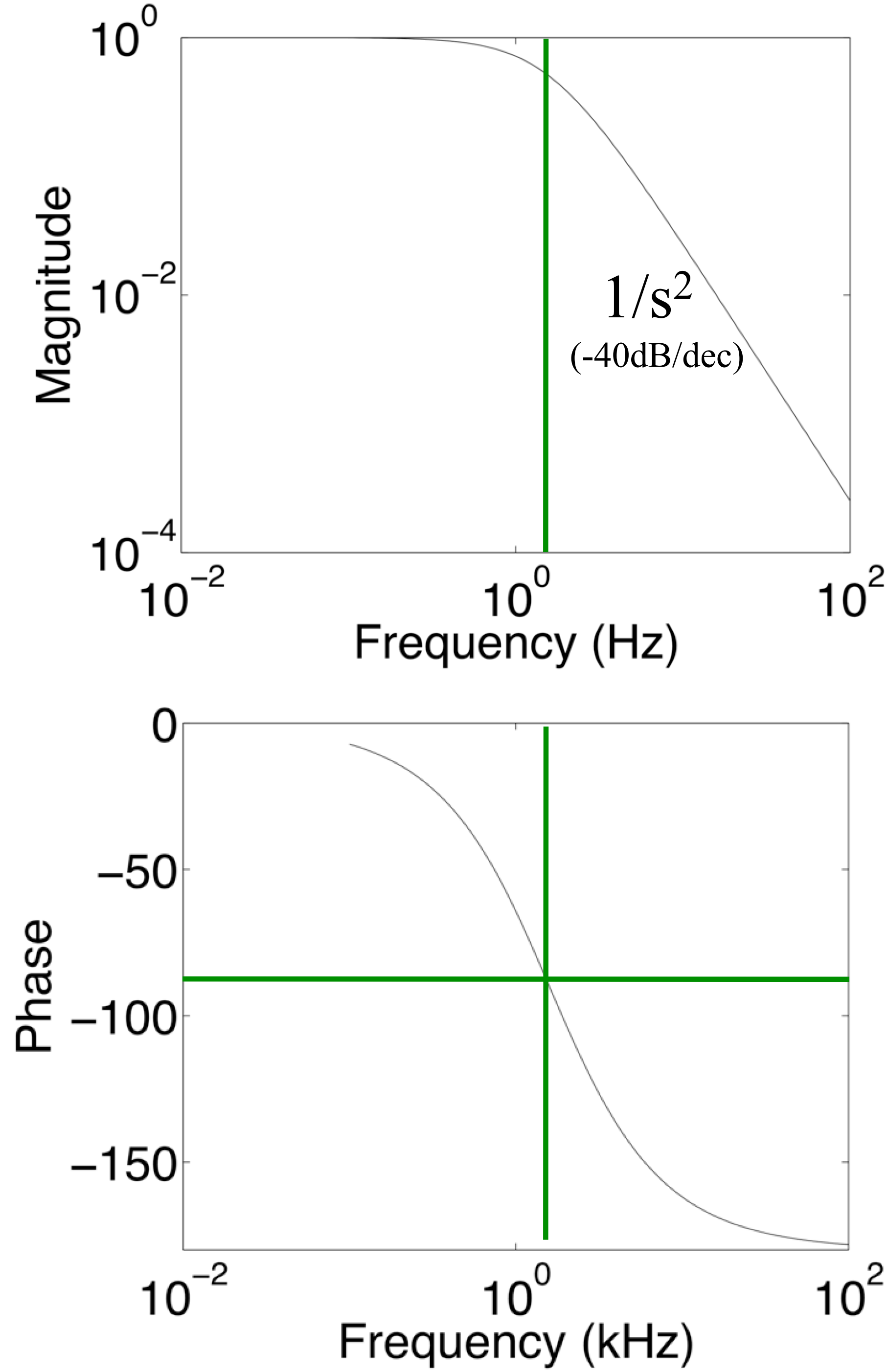
$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{1 + s\tau} \qquad |H(j\omega)| = \frac{1}{\sqrt{1 + \omega^2\tau^2}}$$

$$\text{Phase } = \tan^{-1}(\omega\tau)$$



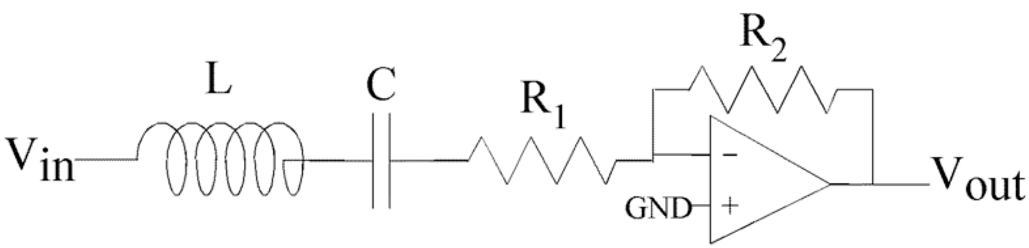
$$H(s) = \frac{1}{1 + s\tau/Q + s^2\tau^2} = \frac{1}{s^2LC + sRC + 1}$$

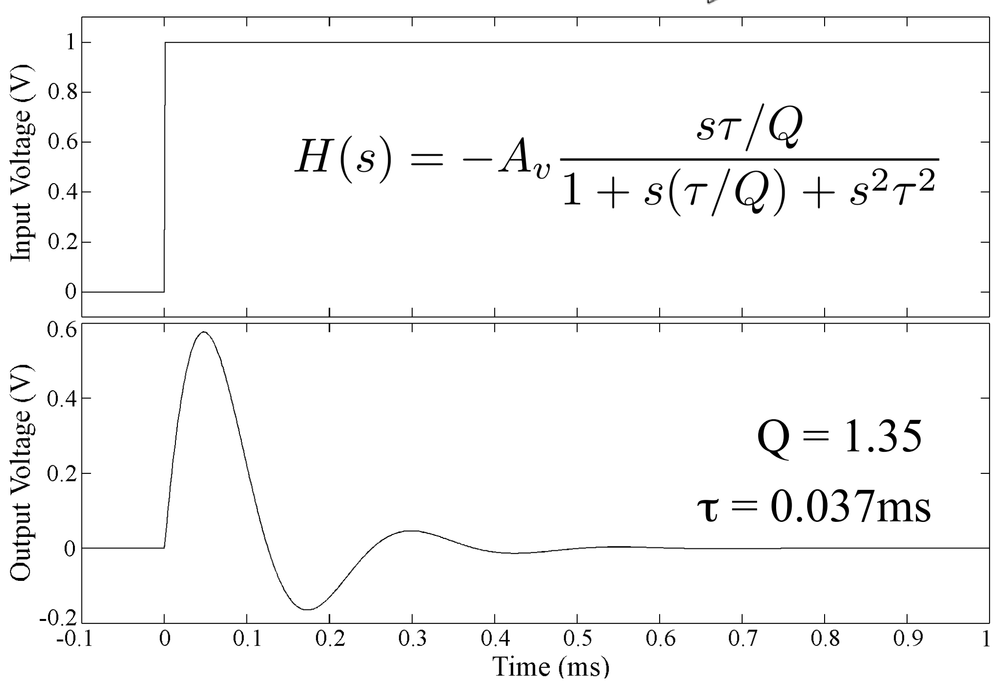


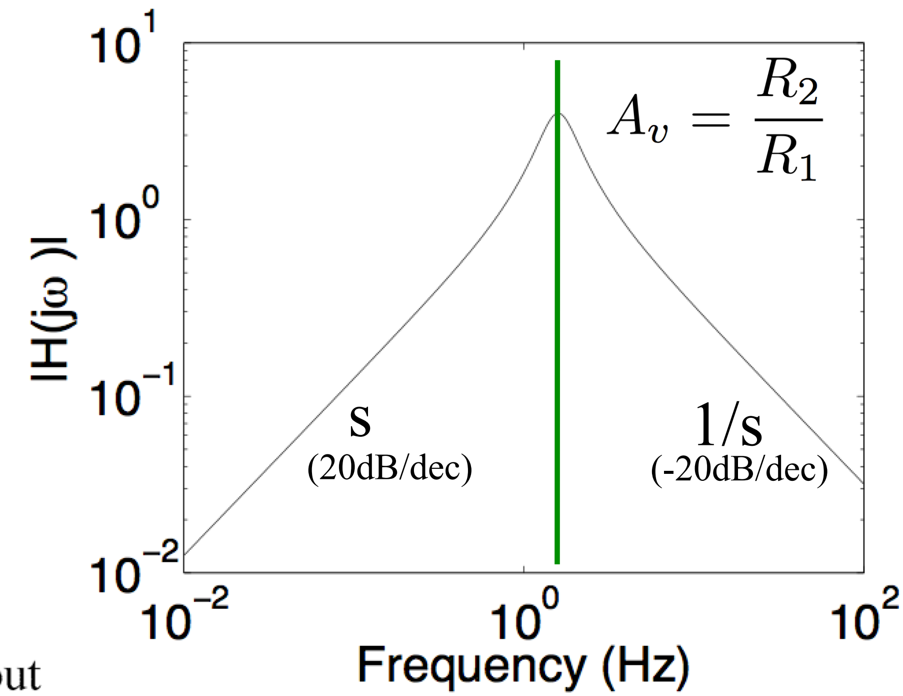


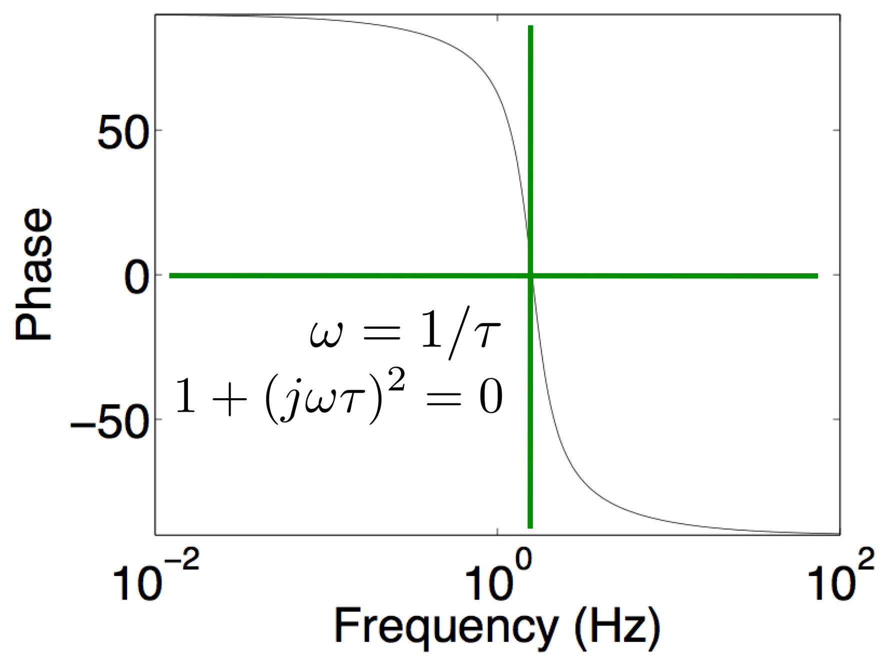
2^{nd}-Order Band-Pass

$$H(s) = -\frac{sR_2C}{1 + sR_1C + s^2LC}$$

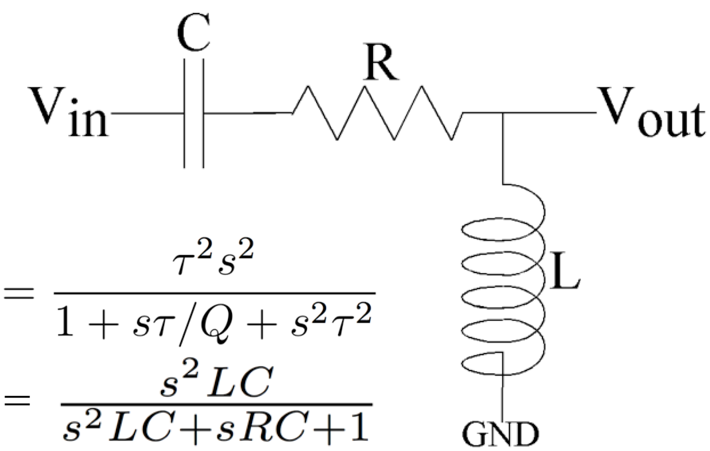




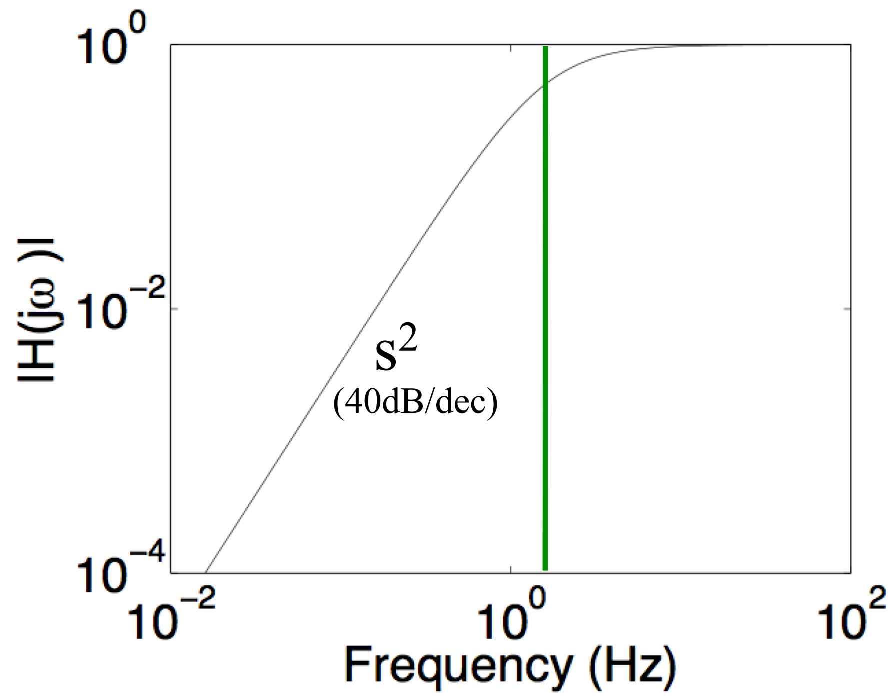


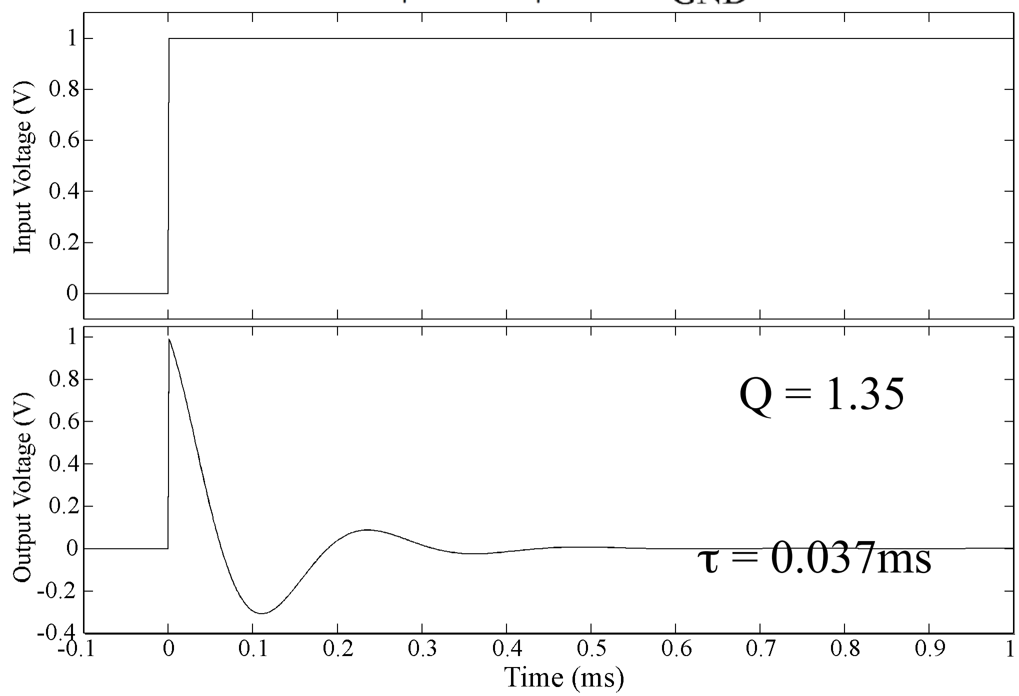


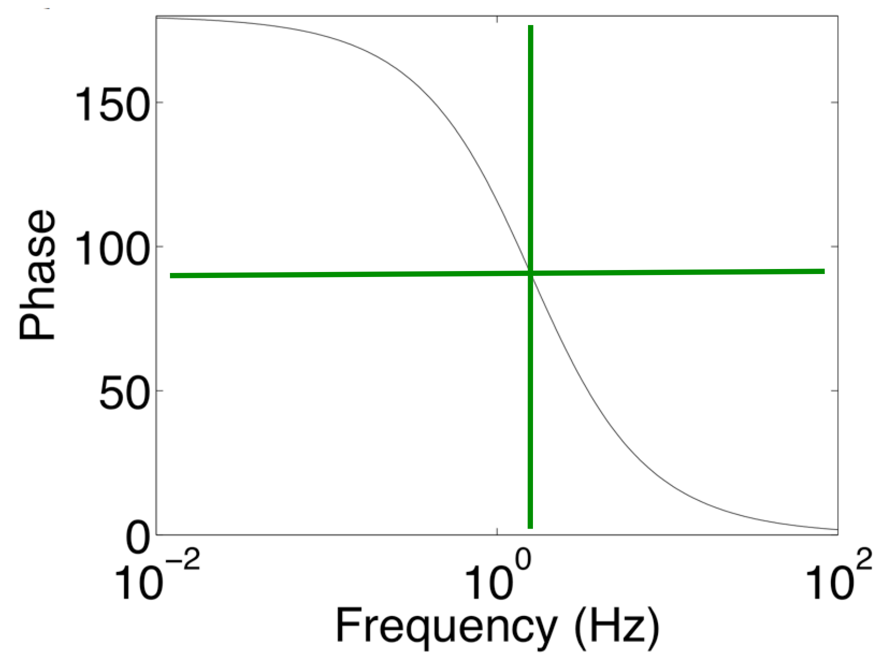
2nd-Order High-Pass

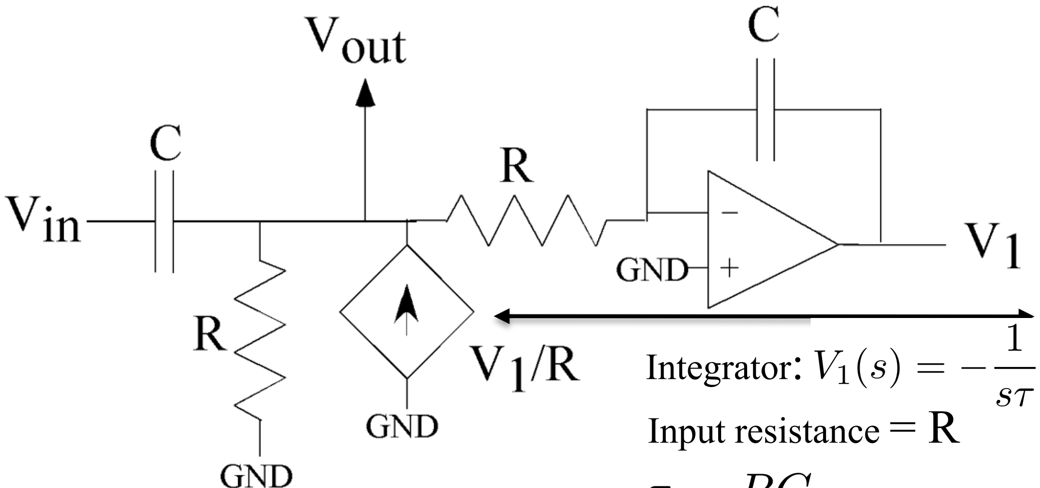


$$H(s) = \frac{\tau^2 s^2}{1 + s\tau/Q + s^2\tau^2} = \frac{s^2 LC}{s^2 LC + sRC + 1}$$





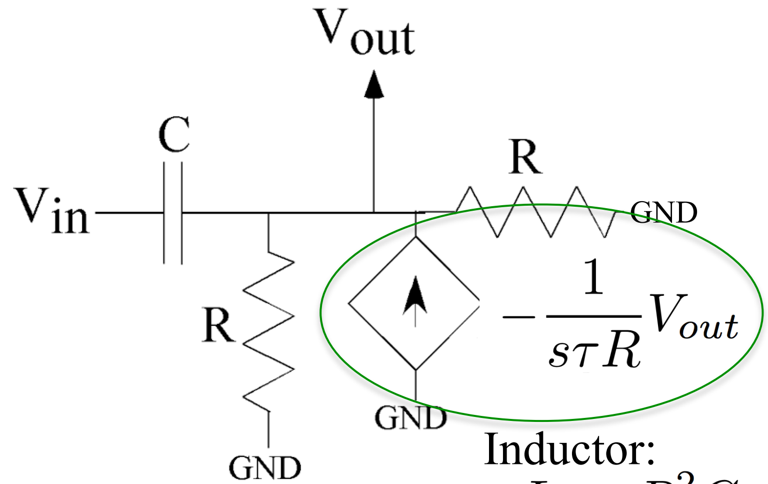




Integrator: $V_1(s) = -\frac{1}{s\tau} V_{out}(s)$

Input resistance = R

$\tau = RC$

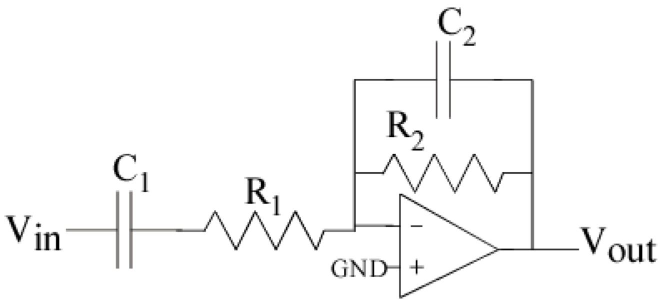


Inductor:
$L \to R^2 C$

$$H(s) = \frac{s^2 LC}{1 + s\frac{2L}{R} + LCs^2}$$

$$H(s) = \frac{\tau^2 s^2}{1 + s\tau/Q + s^2\tau^2}$$

$$H(s) = \frac{s^2\tau^2}{1 + s2\tau + \tau^2 s^2} = \left(\frac{s\tau}{1 + s\tau}\right)^2$$



$R_1 = 10k\Omega$

$R_2 = 100k\Omega$

$C_2 = 10pF$

$C_1 = 1nF$

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = -\frac{R_2 // \frac{1}{sC_2}}{R_1 + \frac{1}{sC_1}}$$

$$= -\frac{1}{(R_1 + \frac{1}{sC_1})(\frac{1}{R_2} + sC_2)}$$

$$= -\frac{sR_2C_1}{(1 + sR_1C_1)(1 + sR_2C_2)}$$

$$= -\frac{R_2}{R_1}\frac{s\tau_1}{(1 + s\tau_1)(1 + s\tau_2)}$$

$$= -\underbrace{\frac{R_2}{R_1}}_{10}\underbrace{\frac{s\tau_1}{(1 + s\tau_1)}}_{\text{HPF}}\underbrace{\frac{1}{(1 + s\tau_2)}}_{\text{LPF}}$$

$\tau_1 = R_1C_1$

$\tau_2 = R_2C_2$

$R_1C_1 > R_2C_2$

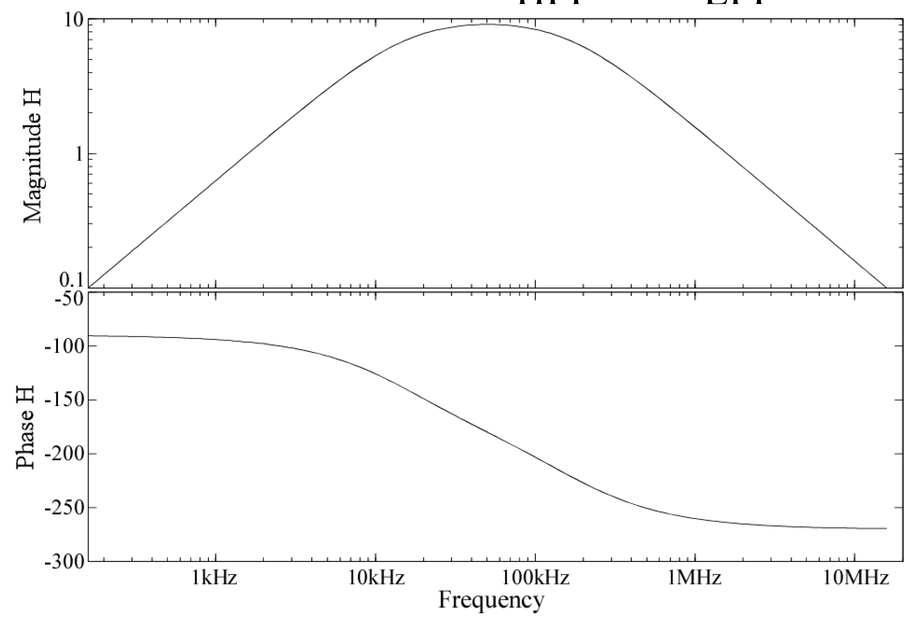
$\tau_1 = 10\mu s$

$(f_1 = 16kHz)$

$\tau_2 = 1\mu s$

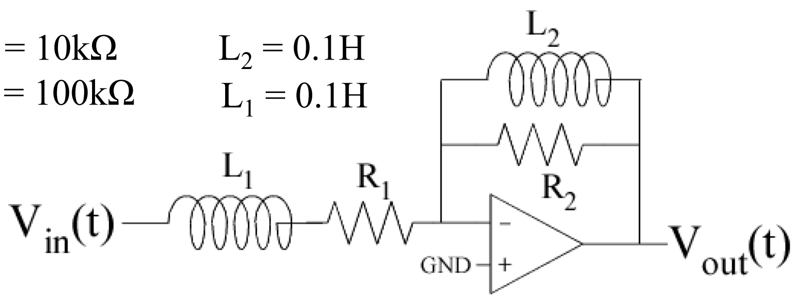
$(f_2 = 160kHz)$

$$H(j\omega) = -\frac{R_2}{R_1}\underbrace{\frac{j\omega\tau_1}{(1 + j\omega\tau_1)}}_{\text{HPF}}\underbrace{\frac{1}{(1 + j\omega\tau_2)}}_{\text{LPF}}$$



$R_1 = 10k\Omega$ $L_2 = 0.1H$

$R_2 = 100k\Omega$ $L_1 = 0.1H$



$$H(s) = -\frac{R2//sL_2}{R_1 + sL_1}$$

$$= -\frac{R_2}{R_1}\frac{s(L_2/R_2)}{1+s(L_2/R_2)}\frac{1}{R_1+sL_1}$$

$$= -\frac{R_2}{R_1}\underbrace{\frac{sL_2}{R_2+sL_2}}_{\text{HPF}}\underbrace{\frac{1}{1+s(L_1/R_1)}}_{\text{LPF}}$$

$\tau_2 = L_1/R_1 = 10\mu s$

$(f_1 = 16kHz)$

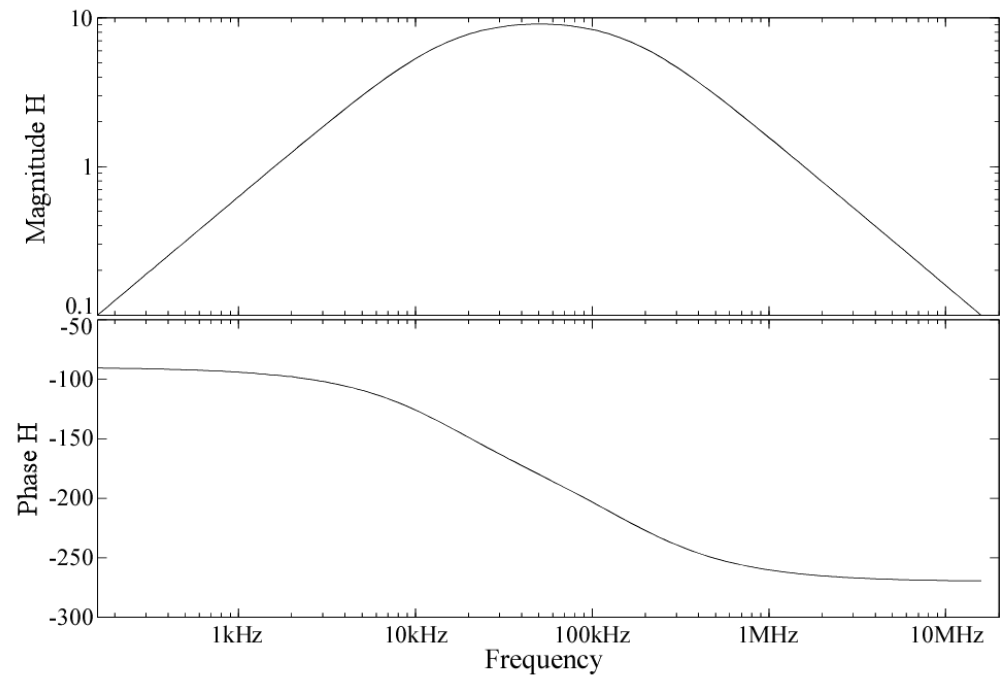
$$= -\frac{R_2}{R_1}\underbrace{\frac{s\tau_1}{(1+s\tau_1)}}_{\text{HPF}}\underbrace{\frac{1}{(1+s\tau_2)}}_{\text{LPF}}$$

$\tau_1 = L_2/R_2 = 1\mu s$

$(f_1 = 160kHz)$

$L_1/R_1 > L_2/R_2$

$$H(j\omega) = -\underbrace{\frac{R_2}{R_1}}_{10}\underbrace{\frac{j\omega\tau_1}{(1+j\omega\tau_1)}}_{\text{HPF}}\underbrace{\frac{1}{(1+j\omega\tau_2)}}_{\text{LPF}}$$

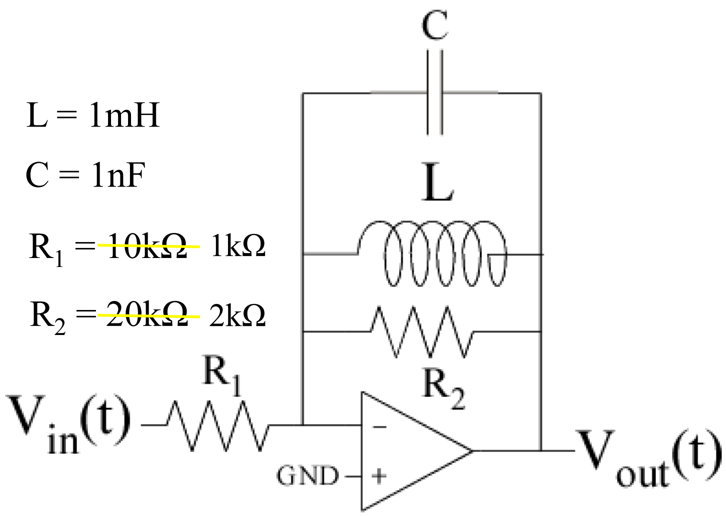


L = 1mH

C = 1nF

R_1 = ~~10kΩ~~ 1kΩ

R_2 = ~~20kΩ~~ 2kΩ



$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = -\frac{R_2//sL//\frac{1}{sC}}{R_1}$$

$$= -\frac{\frac{1}{sC+\frac{1}{R_2}+\frac{1}{sL}}}{R_1}$$

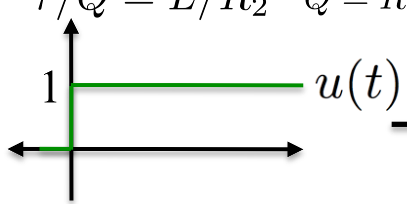
$$= -\frac{R_2}{R_1}\frac{sL}{s^2CLR_2+sL+R_2}$$

$$= -\frac{R_2}{R_1}\frac{s(L/R_2)}{s^2LC+s(L/R_2)+1}$$

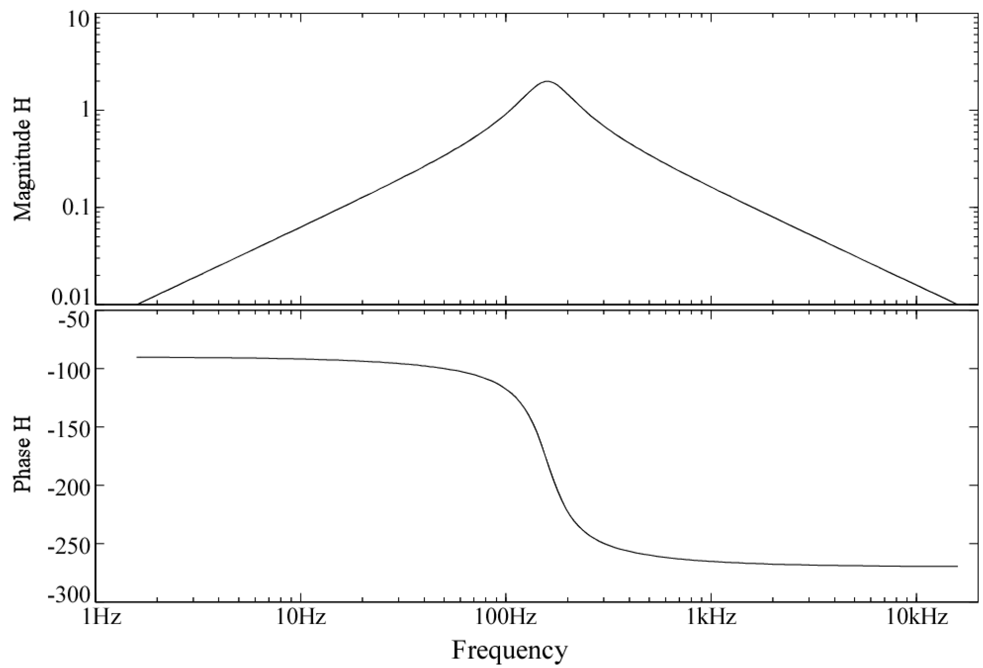
$$= -\frac{R_2}{R_1}\frac{s(\tau/Q)}{s^2\tau^2+s(\tau/Q)+1}$$

τ^2 = LC, τ = 1μs $\quad R_{LC} = \sqrt{L/C}$ = 1kΩ

$\tau/Q = L/R_2 \quad Q = R_2\tau/L = R_2\sqrt{\frac{C}{L}}$ = 2



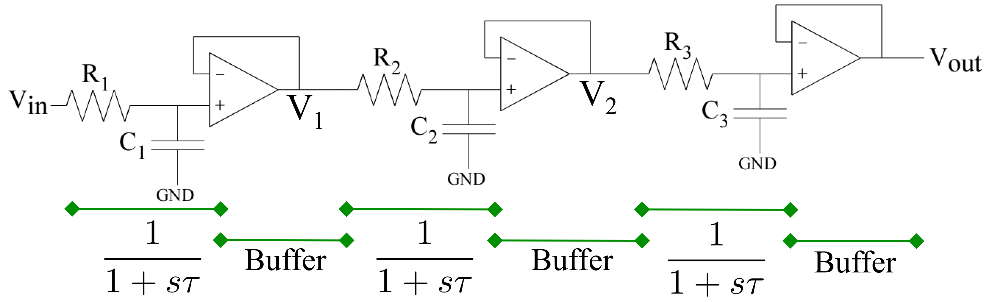
$u(t) \rightarrow \frac{1}{s}$

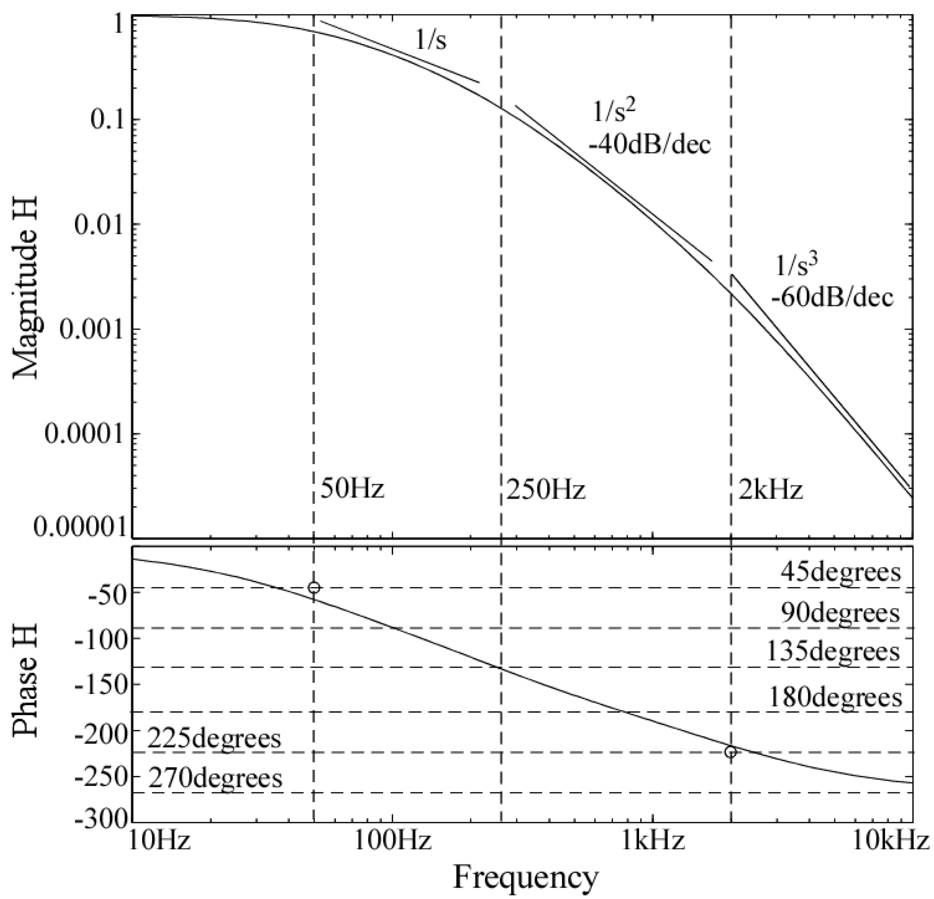


$$V_{out}(s) = -\frac{R_2}{R_1}\frac{1}{Q\tau}\frac{1}{s^2+s\frac{1}{Q\tau}+\frac{1}{\tau^2}}$$

$$e^{-at}\sin(\omega_1 t)u(t) \longleftrightarrow \frac{\omega_1}{(s+a)^2+\omega_1^2}$$

$$V_{out}(t) = -\frac{R_2}{R_1}\frac{1}{\sqrt{1-\frac{1}{4Q^2}}}e^{-t/(2Q\tau)}\sin\left(\frac{t}{\tau}\sqrt{1-\frac{1}{4Q^2}}\right)$$





R_1=1kΩ R_2=3kΩ R_3=4kΩ

C_1=3μF C_2=210nF C_3=20nF

Define: $\tau_1 = R_1 C_1, \tau_2 = R_2 C_2, \tau_3 = R_3 C_3$

τ_1=3.2ms τ_2=637μs τ_3=79.6μs

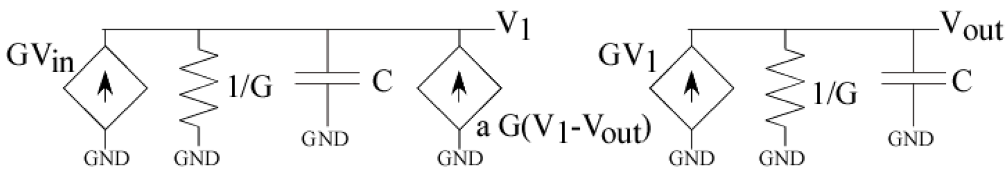
$$H(s) = \frac{1}{(1+s\tau_1)(1+s\tau_2)(1+s\tau_3)}$$

$$H(j\omega) = \frac{1}{(1+j\omega\tau_1)(1+j\omega\tau_2)(1+j\omega\tau_3)}$$

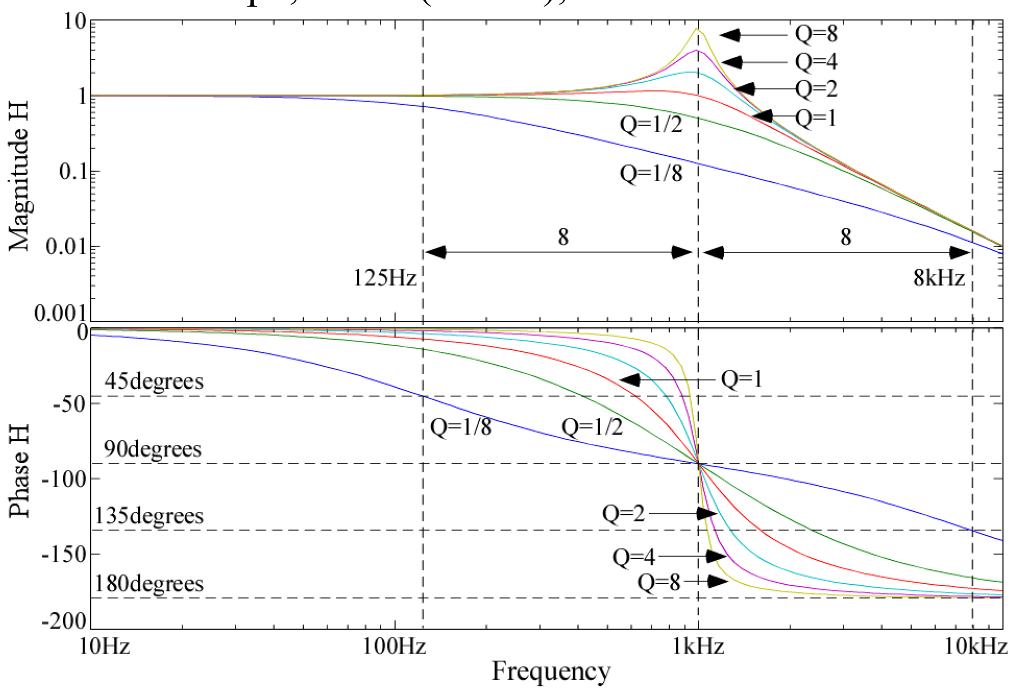
dB of amplitude (x): $20 \log_{10} x$

of power (→ x^2): $10 \log_{10} x^2$
$= 20 \log_{10} x$

(e.g. $P = V^2/R$ for a Resistor)



C = 4pF, G = 1/(40MΩ), C/G = 0.16ms



$$sCV_1 + GV_1 = GV_{in} + aG(V_1 - V_{out})$$

$$sCV_{out} + GV_{out} = GV_1$$

$$\tau = C/G$$

$$(1 - a + s\tau)V_1 = V_{in} - aV_{out}$$

$$(1 + s\tau)V_{out} = V_1$$

$$(1 - a + s\tau)(1 + s\tau)V_{out} = V_{in} - aV_{out}$$

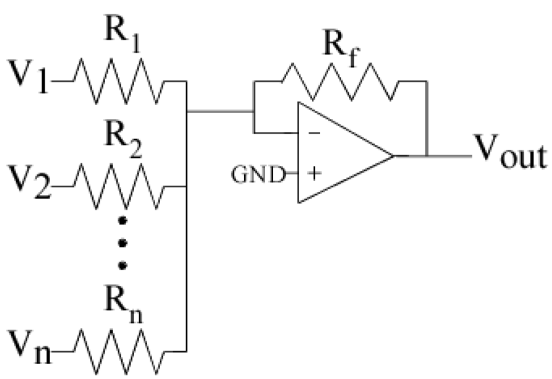
$$[(1 - a + s\tau)(1 + s\tau) + a]V_{out} = V_{in}$$

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{1 + s\tau(2 - a) + s^2\tau^2}$$

$$f_c = \frac{1}{2\pi\tau} \qquad Q = \frac{1}{2 - a}$$

Summation by Current / Charge

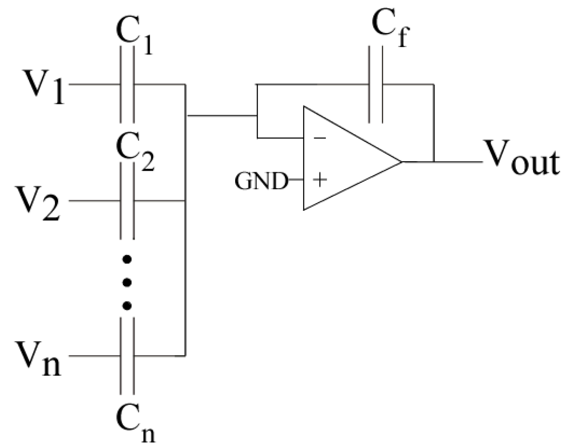
KCL: Sum of currents in = current out



KCL at - Terminal:

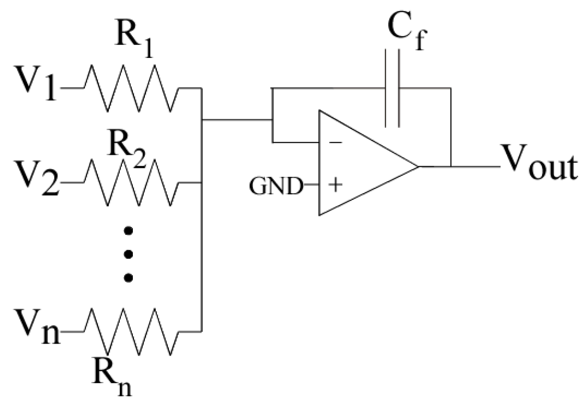
$$\frac{V_1}{R_1} + \frac{V_2}{R_2} + ... + \frac{V_n}{R_n} = -\frac{V_{out}}{R_f}$$

$$V_{out} = -\sum_{k=1}^{n} \frac{R_f}{R_k} V_k$$



$$sC_1V_1 + sC_2V_2 + ... + sC_nV_n = -sC_fV_{out}$$

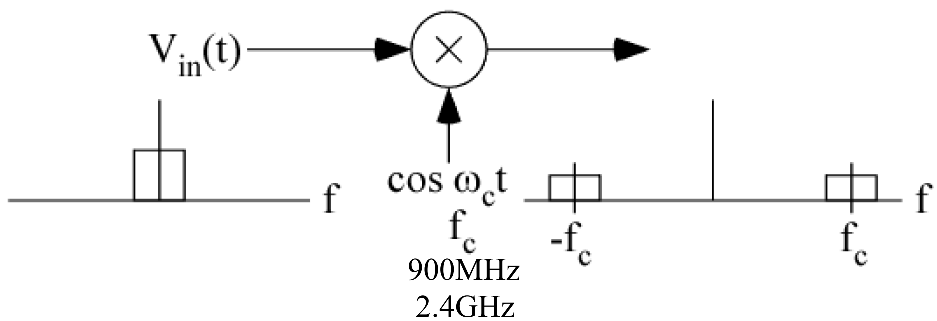
$$V_{out} = V_{offset} - \sum_{k=1}^{n} \frac{C_k}{C_f} V_k$$



$$\frac{V_1}{R_1} + \frac{V_2}{R_2} + ... + \frac{V_n}{R_n} = -sC_fV_{out}$$

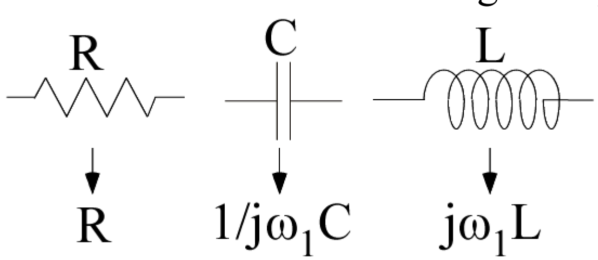
$$V_{out} = -\sum_{k=1}^{n} \frac{1}{sC_fR_k} V_k$$

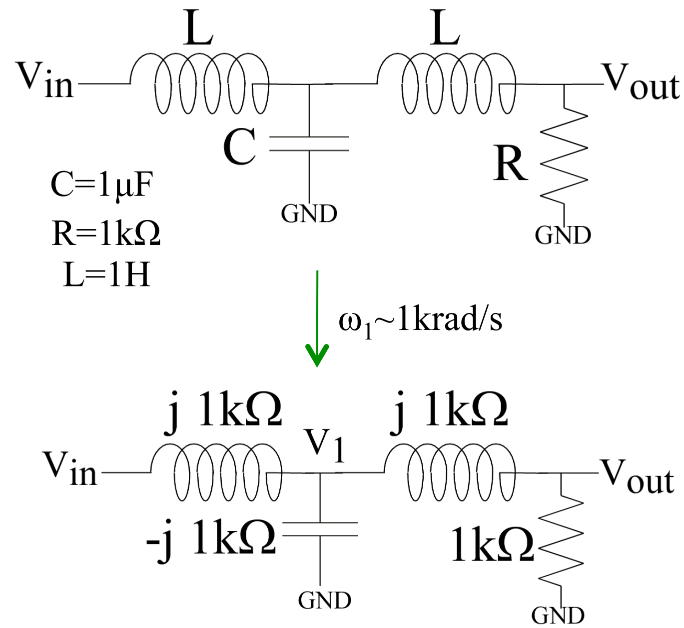
Some circuits operate at a single or near single frequency:
Communications, modulated signals



Another key application: Power Transmission (e.g. 60Hz US)

→ circuit elements: Resistance at a single frequency (f_1, ω_1)





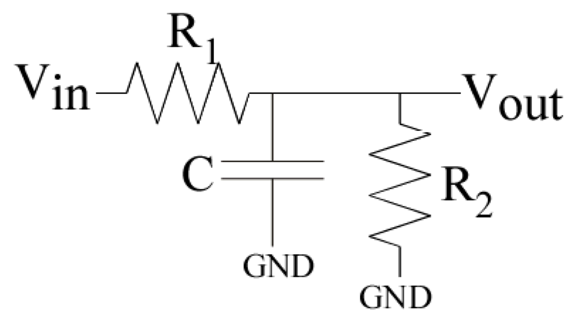
$$\frac{V_{in} - V_1}{j1k\Omega} + \frac{V_{out} - V_1}{j1k\Omega} = \frac{V_1}{-j1k\Omega}$$

$$V_{in} - V_1 + V_{out} - V_1 + V_1 = 0 \rightarrow V_1 = V_{in} + V_{out}$$

$$\frac{V_1 - V_{out}}{j1k\Omega} = \frac{V_{out}}{1k\Omega}$$

$$V_1 = V_{out}(1 + j)$$

$$V_{in} + V_{out} = V_{out}(1 + j) \rightarrow V_{out} = -jV_{in}$$



$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{R_1 // \frac{1}{sC}}{R_1 + R_1 // \frac{1}{sC}} = \frac{R_2}{R_1 + R_2} \frac{1}{1 + s\tau}$$

$$\tau = (R_1 // R_2)C$$

$$|H(j\omega)| = \frac{R_2}{R_1 + R_2} \frac{1}{\sqrt{1 + (\omega\tau)^2}} \quad \text{Phase} : tan^{-1}(\omega\tau)$$

Sinusoidal input → Linear System → Output Sinusoid

freq f_1 — freq f_1

Change in amplitude, phase

$V_{in} \sin \omega_1 t$ → [H(s), H(jω)] → $V_{in} |H(j\omega_1)| \sin(\omega t + \text{phase}(H(j\omega_1)))$

$V_{in} \angle 0$ — $V_{in} |H(j\omega_1)| \angle \text{phase}(H(j\omega_1))$

$V_{in} \sin \omega_1 t$ → $\left[\frac{1}{1 + \tau s}\right]$ → $V_{in} \left|\frac{1}{1 + j\omega_1\tau}\right| \sin(\omega t - \tan^{-1}(\omega_1\tau))$

$V_{in} \angle 0$ — $V_{in} \left|\frac{1}{1 + j\omega_1\tau}\right| \angle \tan^{-1}(\omega_1\tau)$

Phasors: Frequency assumed; just amplitude and phase

Input: $V_{in} \sin \omega_1 t$ → $V_{in} \angle 0$

Output: $V_{in} |H(j\omega_1)| \angle \tan^{-1}\omega_1\tau$

Input is 1V sinusoid, 0 phase

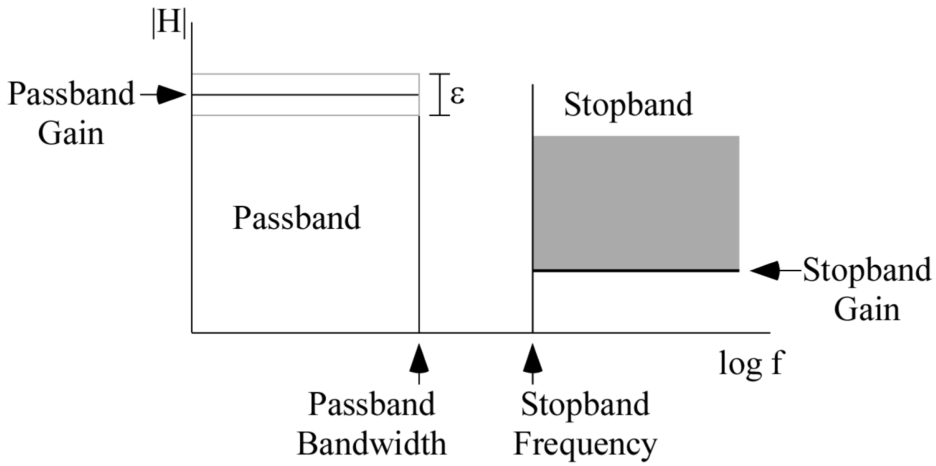
$C = 1\mu F$ $R_1 = 2k\Omega$ $R_2 = 2k\Omega$

$$\tau = 1ms \qquad |H(j\omega)| = \frac{1}{2} \frac{1}{\sqrt{1 + (\omega\tau)^2}}$$

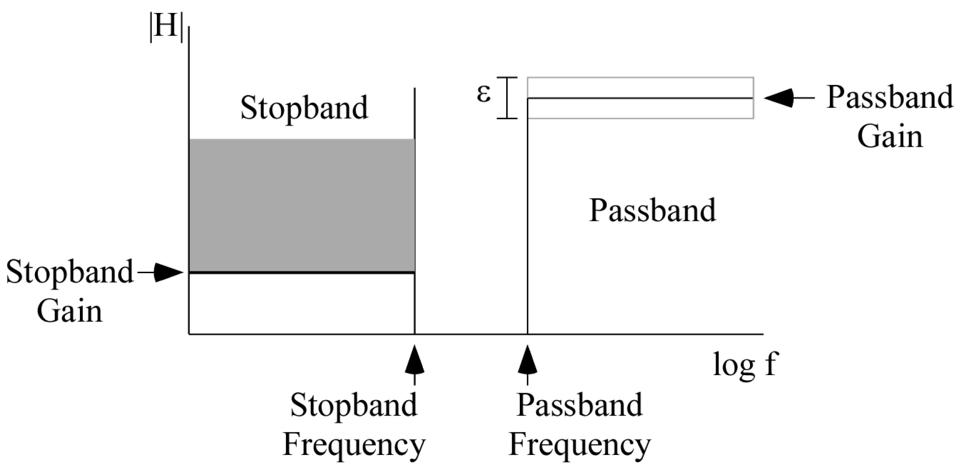
At 160Hz (~1krad/s):

$1V \angle 0$ → $0.354V \angle -45^o$

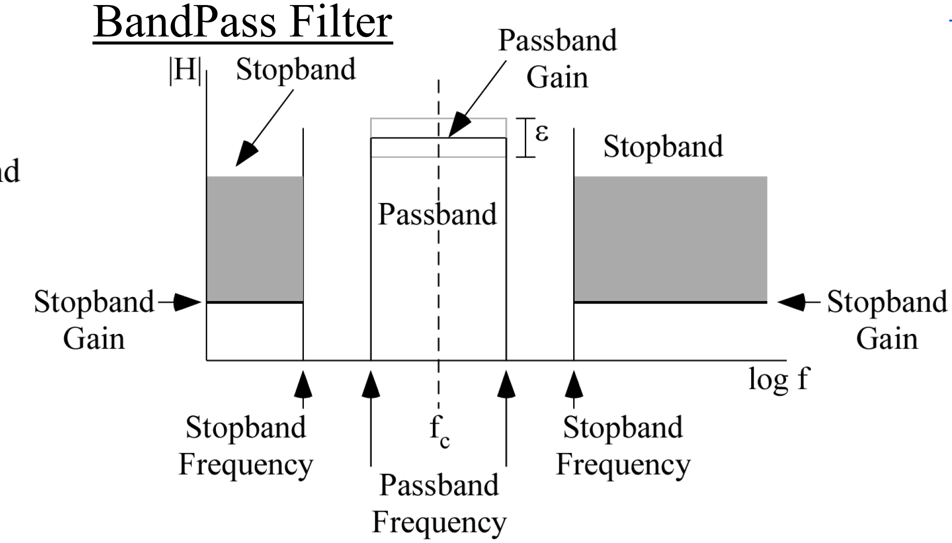
Low-Pass Filter

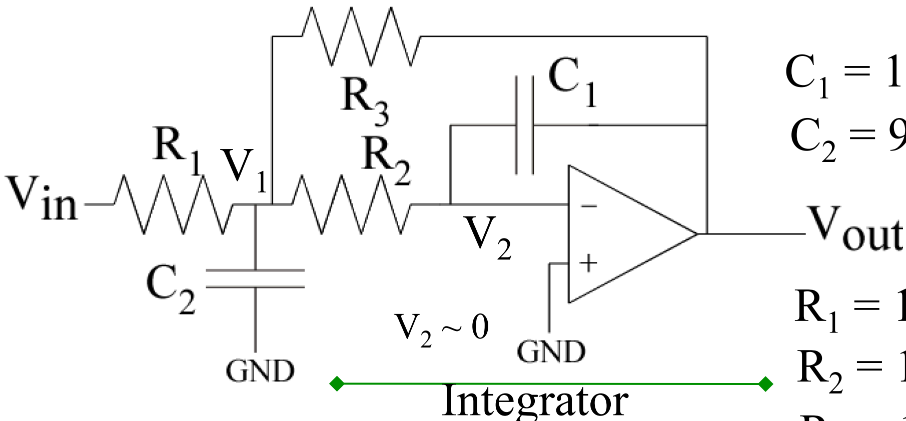


High-Pass Filter



BandPass Filter





$C_1 = 1\text{pF}$

$C_2 = 9\text{pF}$

$R_1 = 1\text{M}\Omega$

$R_2 = 1\text{M}\Omega$

$R_3 = 1\text{M}\Omega$

$$V_{out} = -V_1 \frac{1}{sR_2C_1} \rightarrow -sR_2C_1V_{out} = V_1$$

$$sC_2V_1 = \frac{V_{in} - V_1}{R_1} + \frac{V_{out} - V_1}{R_3} - \frac{V_1}{R_2}$$

$$(sC_2 + \frac{1}{R_1//R_2//R_3})V_1 = \frac{V_{in}}{R_1} + \frac{V_{out}}{R_3}$$

$$-(sC_2 + \frac{1}{R_1//R_2//R_3})sR_2C_1V_{out} = \frac{V_{in}}{R_1} + \frac{V_{out}}{R_3}$$

$$\left(s^2R_2R_3C_1C_2 + s\frac{R_3R_2}{R_1//R_2//R_3}C_1 + 1\right)V_{out} = -\frac{R_3}{R_1}V_{in}$$

$$\left((s^2R_2R_3C_1C_2 + s\left(R_2 + R_3 + \frac{R_2R_3}{R_1}\right)C_1 + 1\right)V_{out} = -\frac{R_3}{R_1}V_{in}$$

$$\tau^2 = R_2R_3C_1C_2 \quad \text{gain} = -\frac{R_3}{R_1}$$

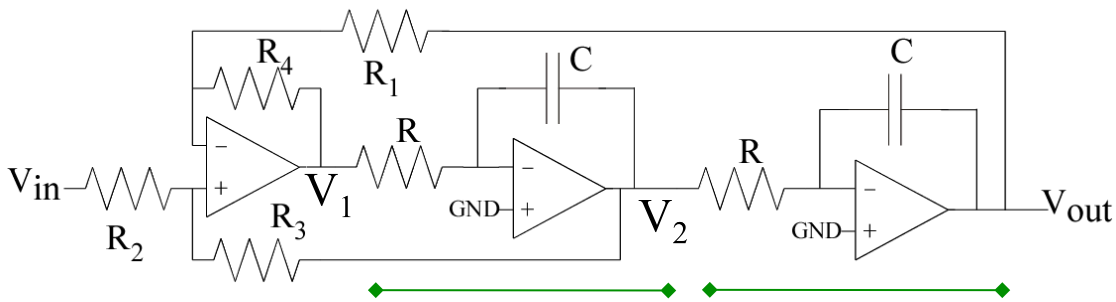
$$\frac{\tau}{Q} = \left(R_2 + R_3 + \frac{R_2R_3}{R_1}\right)C_1$$

$$Q = \frac{\tau}{\left(R_2 + R_3 + \frac{R_2R_3}{R_1}\right)C_1}$$

$$= \sqrt{\frac{C_2}{C_1}}\frac{\sqrt{R_2R_3}}{\left(R_2 + R_3 + \frac{R_2R_3}{R_1}\right)}$$

$$\tau = 3\mu s \quad \text{gain} = -1 \quad Q = 1$$

Rauch Biquadratic cell



Integrator Integrator

$C_2 = 1\text{nF}$

$$V_2 = -V_1 \frac{1}{sRC} \qquad V_{out} = -V_2 \frac{1}{sRC}$$

$$V_1 = -\frac{R_4}{R_1} V_{out} + \frac{R_4 + R_1}{R_1} \left(\frac{R_3 V_{in}}{R_3 + R_2} + \frac{R_2 V_2}{R_2 + R_3} \right)$$

$$s^2 (RC)^2 V_{out} = -\frac{R_4}{R_1} V_{out} + \frac{R_4 + R_1}{R_1} \left(\frac{R_3 V_{in}}{R_3 + R_2} - \frac{R_2 s(RC) V_{out}}{R_2 + R_3} \right)$$

$$\left(s^2 (RC)^2 + \frac{R_4 + R_1}{R_1} \frac{s(RC) R_2}{R_2 + R_3} + \frac{R_4}{R_1} \right) V_{out} = \frac{R_4 + R_1}{R_1} \left(\frac{R_3 V_{in}}{R_3 + R_2} \right)$$

$$\left(s^2 (RC)^2 R_1 (R_2 + R_3) + s(RC) R_2 (R_4 + R_1) + R_4 (R_2 + R_3) \right) V_{out} = (R_4 + R_1) R_3 V_{in}$$

$$\frac{V_{out}}{V_{in}} = \frac{(R_4 + R_1) R_3}{s^2 (RC)^2 R_1 (R_2 + R_3) + s(RC) R_2 (R_4 + R_1) + R_4 (R_2 + R_3)}$$

(LPF)

$$\tau = RC \sqrt{\frac{R_1}{R_4}} \qquad \text{Gain} = \frac{R_4 + R_1}{R_4} \frac{R_3}{R_2 + R_3}$$

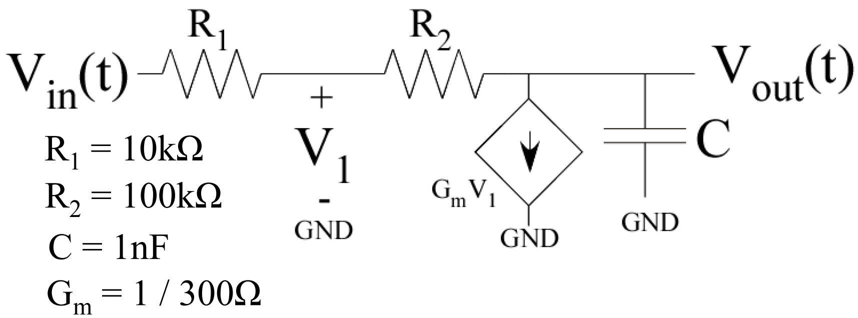
Kerwin-Huelsman-Newcomb Circuit

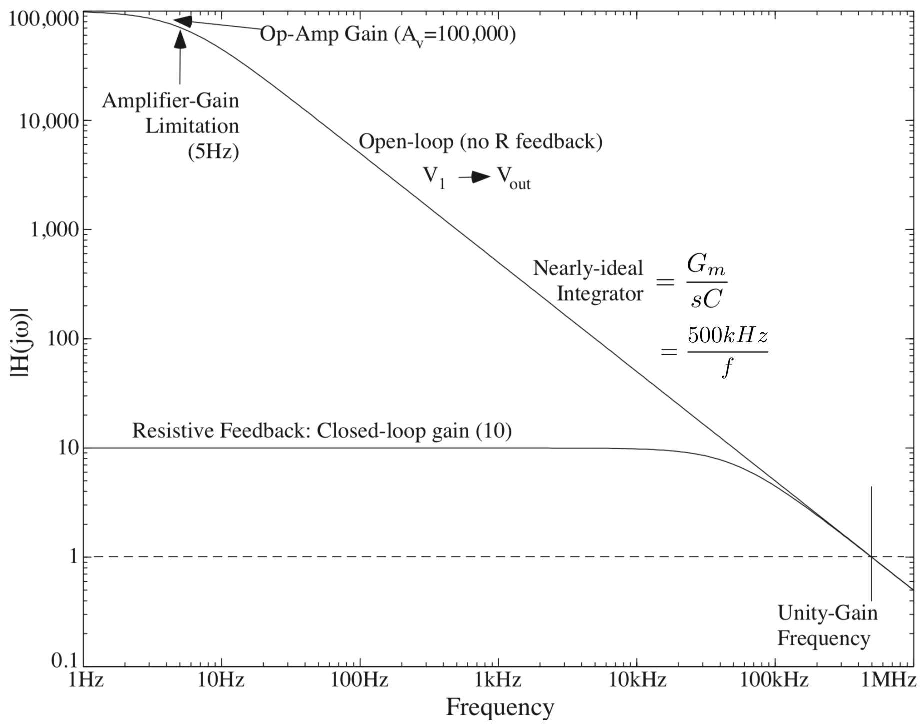
V_2 → BPF V_1 → HPF

$$\frac{\tau}{Q} = RC \frac{R_2}{R_2 + R_3} \frac{R_4 + R_1}{R_4}$$

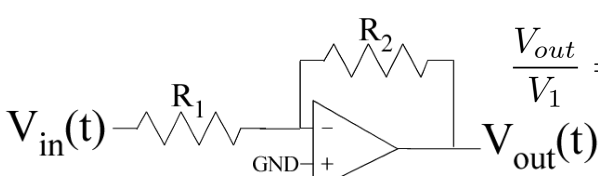
$$Q = \frac{RC \sqrt{\frac{R_1}{R_f}}}{RC \frac{R_2}{R_2 + R_3} \frac{R_4 + R_1}{R_4}} = \frac{R_2 + R_3}{R_2} \frac{\sqrt{R_4 R_1}}{R_4 + R_1}$$

R_1	R_2	R_3	R_4	R	τ	Q
1kΩ	1kΩ	1kΩ	1kΩ	1kΩ	1μs	1
4kΩ	1kΩ	1kΩ	4kΩ	1kΩ	1μs	1
1kΩ	4kΩ	4kΩ	1kΩ	1kΩ	1μs	1
1kΩ	1kΩ	1kΩ	1kΩ	4kΩ	4μs	1
4kΩ	1kΩ	1kΩ	1kΩ	1kΩ	2μs	4/5
1kΩ	1kΩ	1kΩ	4kΩ	1kΩ	0.5μs	4/5





Ideal Op-Amp: Output capacitance has no effect



$$\frac{V_{out}}{V_1} = -\frac{R_2}{R_1}$$

Real Op-Amps can have a frequency response

$$\frac{V_{in} - V_1}{R_1} = \frac{V_1 - V_{out}}{R_2} = G_m V_1 + sCV_{out}$$

$$\frac{V_{in}}{R_1} + \frac{V_{out}}{R_2} = \frac{V_1}{R_1//R_2}$$

$$R_2 V_{in} + R_1 V_{out} = (R_1 + R_2) V_1$$

$$\left(G_m - \frac{1}{R_2}\right) V_1 + \left(sC + \frac{1}{R_2}\right) V_{out} = 0$$

As $G_m >> \frac{1}{R_2}$,

$$G_m V_1 + \left(sC + \frac{1}{R_2}\right) V_{out} = 0$$

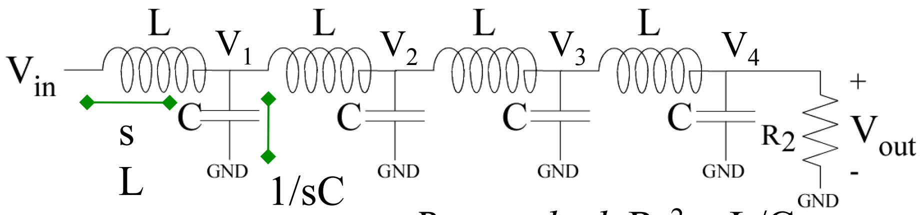
$$G_m R_2 V_1 + (sCR_2 + 1)\, V_{out} = 0$$

$$V_{out} = -\frac{G_m R_2}{1 + sCR_2} V_1 \longrightarrow = -\frac{G_m}{sC} V_1$$

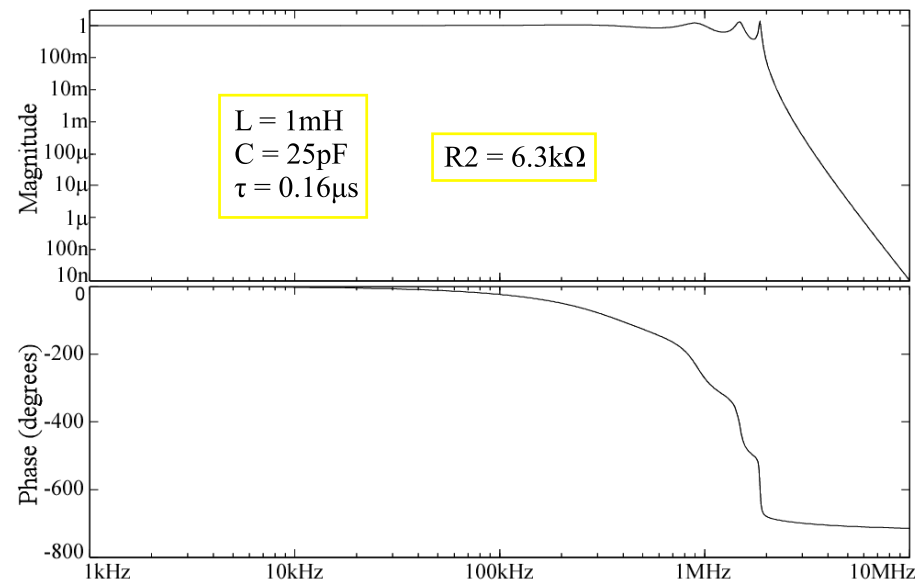
$$\frac{V_{out}}{V_1} = -\frac{R_2}{R_1} \frac{1}{1 + \frac{1}{G_m}\frac{R_1+R_2}{R_2 R_1} + s\frac{C}{G_m}\frac{R_1+R_2}{R_1}}$$

10   0.033 (small)   ~50kHz

$$\frac{V_4}{V_{in}} = \frac{1}{s^8\tau^8 + s^7\tau^7 + 7s^6\tau^6 + 6s^5\tau^5 + 15s^4\tau^4 + 10s^3\tau^3 + 10s^2\tau^2 + 4s\tau + 1}$$



R_2 matched: $R_2^2 = L/C$



$$sCV_1 = \frac{V_{in} - V_1}{sL} + \frac{V_2 - V_1}{sL}$$

$$s^2LCV_1 = V_{in} - 2V_1 + V_2$$

$$\tau^2 = LC$$

$$s^2\tau^2 V_1 = V_{in} - 2V_1 + V_2$$

$$(s^2\tau^2 + 2)V_1 = V_{in} + V_2$$

$$(s^2\tau^2 + 2)V_2 = V_1 + V_3$$

$$(s^2\tau^2 + 2)V_3 = V_2 + V_4$$

$$sCV_4 + \frac{V_4}{R_2} = \frac{V_3 - V_4}{sL}$$

$$(s^2LC + s\frac{L}{R_2} + 1)V_4 = V_3$$

from matching:

$$(s^2\tau^2 + s\tau + 1)V_4 = V_3$$

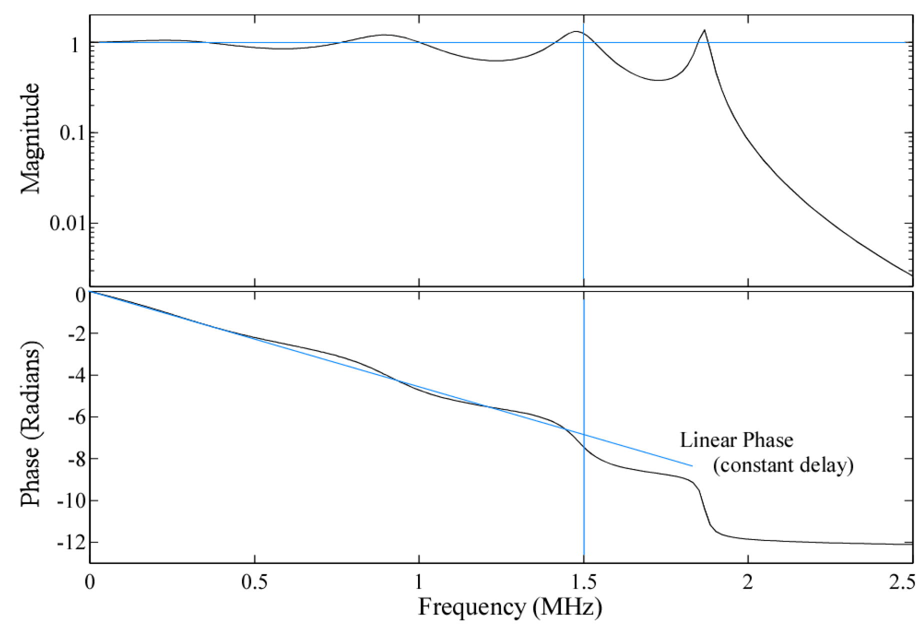
$$(s^2\tau^2+2)^2V_2 = (s^2\tau^2+2)V_1+(s^2\tau^2+2)V_3$$
$$= V_{in}+2V_2+V_4$$

$$\left[(s^2\tau^2 + 2)^2 - 2\right] V_2 = V_{in} + V_4$$

$$(s^2\tau^2 + 2)(s^2\tau^2 + s\tau + 1)V_4 = V_2 + V_4$$

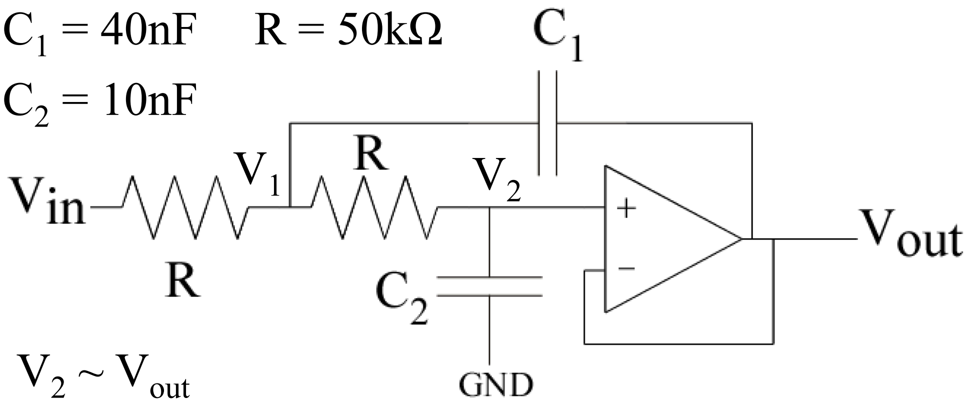
$$\left[(s^2\tau^2 + 2)(s^2\tau^2 + s\tau + 1) - 1\right] V_4 = V_2$$

$$\left(\left[(s^2\tau^2 + 2)^2 - 2\right] \left[(s^2\tau^2 + 2)(s^2\tau^2 + s\tau + 1) - 1\right] - 1\right) V_4 = V_{in}$$



$C_1 = 40nF \quad R = 50k\Omega$

$C_2 = 10nF$



$V_2 \sim V_{out}$

$$\frac{V_1 - V_{out}}{R} = sC_2V_2 \rightarrow \tau_2 = RC_2, (1 + s\tau_2)V_{out} = V_1$$

$$\frac{V_{in} - V_1}{R} + \frac{V_{out} - V_1}{R} = sC_1(V_1 - V_{out})$$

$$\tau_1 = RC_1, V_{in} + (1 + s\tau_1)V_{out} = (2 + s\tau_1)V_1$$

$$V_{in} = \left((2 + s\tau_1)(1 + s\tau_2) - (1 + s\tau_1)\right) V_{out}$$

$$\frac{V_{out}}{V_{in}} = \frac{1}{s^2\tau_1\tau_2 + 2s\tau_2 + s\tau_1 + 2 - 1 - s\tau_1}$$

$$= \frac{1}{s^2\tau_1\tau_2 + 2s\tau_2 + 1}$$

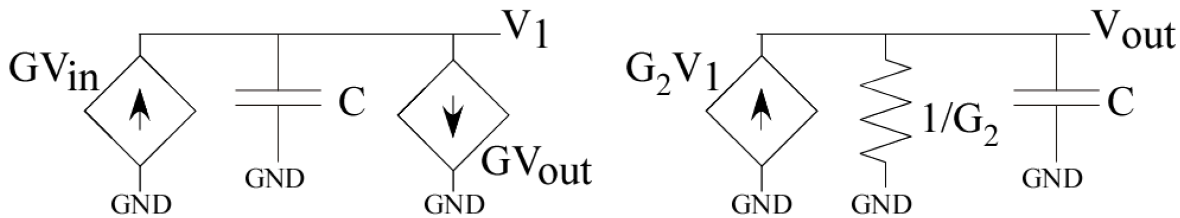
$$\tau^2 = \tau_1\tau_2 \qquad \frac{\tau}{Q} = 2\tau_2, Q = \frac{\tau}{2\tau_2} = \frac{1}{2}\sqrt{\frac{C_1}{C_2}}$$
$$= R\sqrt{C_1C_2}$$

$\tau_1 = 2ms \qquad \tau = 1ms$

$\tau_2 = 0.5ms \qquad Q = 1$

Sallen & Key 2nd Order Circuit

C_1	C_2	R	τ	Q
40nF	10nF	50kΩ	1ms	1
80nF	5nF	50kΩ	1ms	2
40nF	10nF	100kΩ	2ms	1
80nF	20nF	50kΩ	2ms	1



$$sCV_1 = G(V_{in} - V_{out}) \rightarrow \tau_1 = \frac{C}{G}, s\tau_1 V_1 = V_{in} - V_{out}$$

$$sCV_{out} + G_2 V_{out} = G_2 V_1 \rightarrow \tau_2 = \frac{C}{G_2}, (1 + s\tau_2)V_{out} = V_1$$

$$H(s) = \frac{V_{out}}{V_{in}} = \frac{1}{1 + s\tau_1 + s^2 \tau_1 \tau_2}$$

$$\tau^2 = \tau_1 \tau_2 \qquad \tau_1 = \frac{\tau}{Q} \qquad Q = \frac{\tau}{\tau_1} = \sqrt{\frac{\tau_2}{\tau_1}} \text{ = sqrt(G/G2)}$$

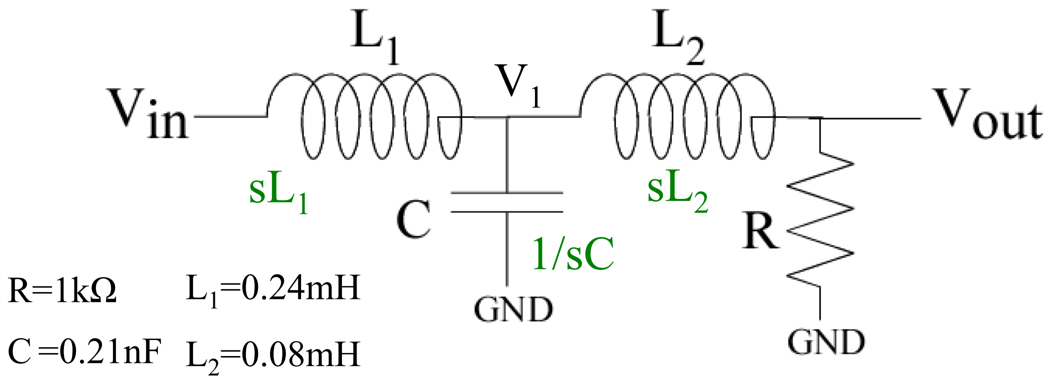
$C = 1\text{pF}$ $\quad$ $G = 1 / 4\text{M}\Omega$

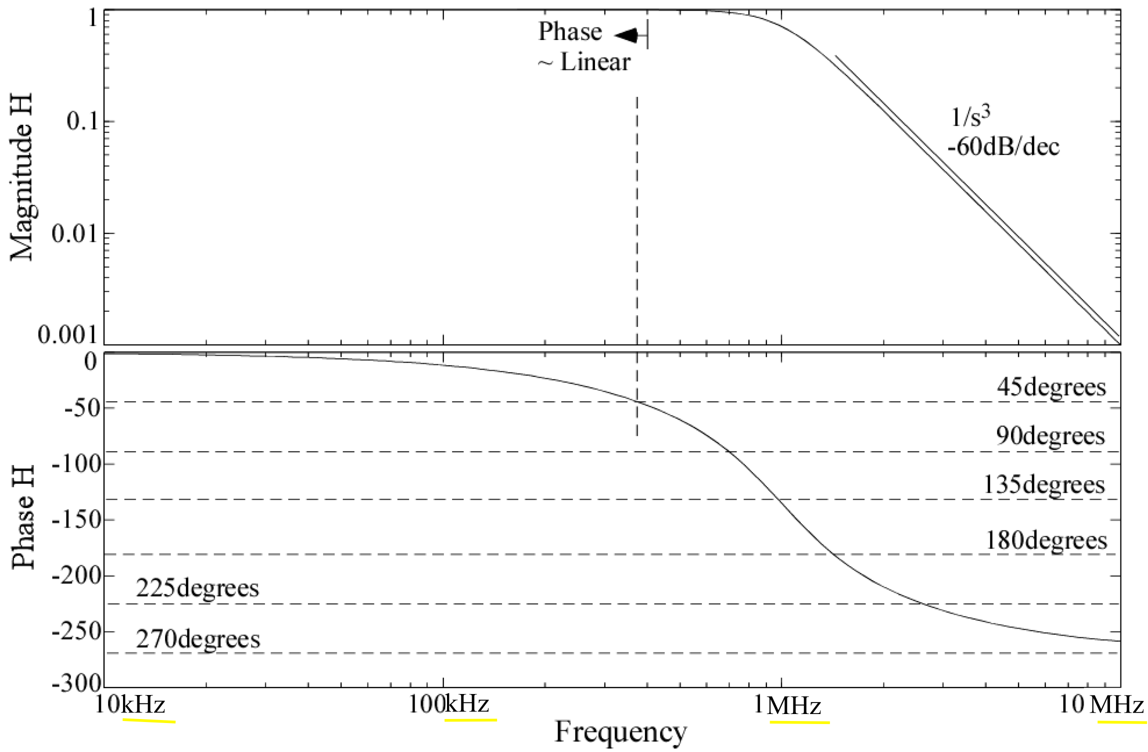
$G_2 = 1 / 1\text{M}\Omega$

$\tau_1 = 4\mu\text{s}$ $\quad$ $\tau = 2\mu\text{s}$

$\tau_2 = 1\mu\text{s}$ $\quad$ $Q = 1/2$

Circuit: One form of Tau-Thomas; also called Diff2 structure





$$\frac{V_1 - V_{out}}{sL_2} = V_{out}/R$$

$$\rightarrow V_1 = V_{out}\left(1 + s\frac{L_2}{R}\right)$$

$$\frac{V_{in} - V_1}{sL_1} + \frac{V_{out} - V_1}{sL_2} = sCV_1$$

$$\frac{V_{in}}{sL_1} + \frac{V_{out}}{sL_2} = \left(sC + \frac{1}{sL_1} + \frac{1}{sL_2}\right) V_1$$

$$V_{in} = sL_1\left(\left(sC + \frac{1}{sL_1} + \frac{1}{sL_2}\right)\left(1 + s\frac{L_2}{R}\right) - \frac{1}{sL_2}\right) V_{out}$$

$$V_{in} = \frac{1}{R}\left(s^3 L_1 L_2 C + s^2 L_1 R C + s(L_1 + L_2) + R\right) V_{out}$$

With values: $\tau \sim 0.16\mu s$

$$\frac{V_{out}}{V_{in}} = \frac{1}{s^3\tau^3 + 2s^2\tau^2 + 2s\tau + 1}$$

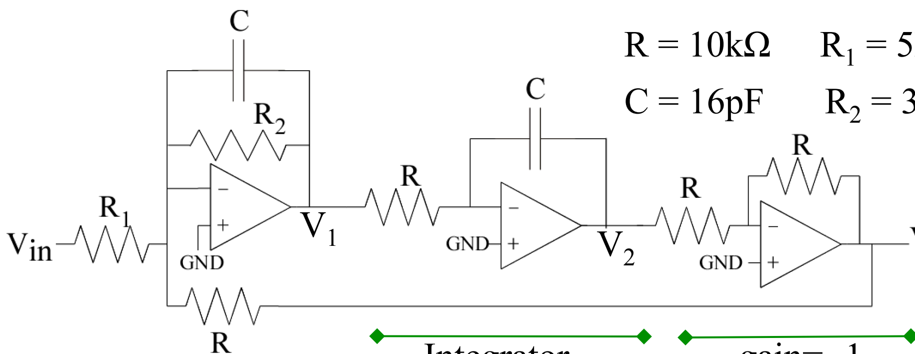
$$= \frac{1}{(s\tau + 1)(s^2\tau^2 + s\tau + 1)}$$

Q=1

3rd Order Butterworth Filter

R = 10kΩ　　R_1 = 5kΩ

C = 16pF　　R_2 = 30kΩ



LPF block

$$\frac{V_{in}}{R_1} + \frac{V_{out}}{R} = -(sC + \frac{1}{R_2})V_1$$

Integrator

$$V_2 = -V_1 \frac{1}{sRC}$$

gain= -1

V_{out} = -V_2

$$\frac{1}{R_1}V_{in} = -V_{out}\left(\frac{1}{R} + (sC + \frac{1}{R_2})sRC\right)$$

$$\frac{R}{R_1}V_{in} = -V_{out}\left(1 + s\tau\frac{R}{R_2} + s^2\tau^2\right)$$

$$\tau = RC$$

$$Q = \frac{R_2}{R}$$

$$\frac{V_{out}}{V_{in}} = -\frac{R}{R_1}\frac{1}{1 + s\tau\frac{R}{R_2} + s^2\tau^2}$$

V_1 → BPF

V_{out} → LPF

Tow-Thomas Second-Order Configuration